# Artin Formalism, for Kleinian Groups, via Heat Kernel Methods

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**Abstract**

Let $(\Gamma', \psi)$, $(\Gamma, \pi)$ be pairs consisting of a cofinite Kleinian group (discrete subgroup of $\mathrm{SL}_2(\mathbf{C})$) and a finite dimensional unitary representation of the group. Suppose that $\Gamma$ and $\Gamma'$ are commensurable and that $\pi$ and $\psi$ are related by induction of representations. We show that each term in the Selberg trace formula associated to $(\Gamma', \psi)$ is equal to the corresponding term in the trace formula associated to $(\Gamma, \pi)$. Our result generalizes certain methods and results of Venkov and Zograf in [20], [21], and Friedman in [9]. As a corollary of our result on the Selberg trace formula, we obtain a generalization of the previously known cases of Artin Formalism of Selberg zeta functions. Our approach to the problem lies much closer to that of Jorgenson and Lang in [12] (where certain higher-dimensional co-compact cases are treated), as opposed to [21] and [9]. This is because we show the equality of the sections of the vector bundles on taking a 'partial trace' of each, *i.e.*, before tracing down to functions and taking the integral over the diagonal in $\Gamma\backslash G \times \Gamma\backslash G$, as one eventually does in the trace formula. In addition to giving a more transparent proof that allows one to dispense with a normality assumption found in previous treatments, our method seems to have better prospects for unifying the various aspects of 'inductivity' of related spectral objects, such as the Selberg zeta function and the determinant of the scattering matrix. The method appears to generalize to certain higher dimensional situations.

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# 1 Introduction

**Generalities.** Let $\mathbf{H}^n$ be the upper half space of dimension $n$. This can be realized as the quotient $H = G/K$, where $G = \mathrm{SO}(n,1)$ and $K = \mathrm{SO}(n)$.

Let $\Gamma$ a discrete subgroup of isometries such that $\Gamma\backslash\mathbf{H}^n$ has finite volume (with respect to the $G$-invariant measure).

The datum of a finite dimensional unitary representation $(\pi, W)$ of $\Gamma$ is equivalent to the datum of a flat unitary vector bundle $\mathcal{H}_\pi$ over the locally symmetric space $X_\Gamma = \Gamma\backslash\mathbf{H}$. The bundle is obtained as

$$\mathcal{H}_\pi = (\mathbf{H} \times W)/\sim,$$

with $\sim$ the equivalence relation defined in (8), below.

Associated to such a flat vector bundle (or to such a representation), one can associate the following objects

(a) $L^2(\Gamma\backslash\mathbf{H}, \pi)$ the Hilbert space of $L^2$ sections of $\mathcal{H}_\pi$. Also, the Laplace operator $\Delta$ acting on $L^2(\Gamma\backslash\mathbf{H}, \pi)$ (or more precisely a certain dense subspace of the $L^2$ subpsace). For an appropriate definition of $L^2$, as well as the extension of $\Delta$ (to a self-adjoint unbounded operator), we refer to Chapter 4 of [6] and Chapter 4 (p. 17) of [22].

(b) Selberg zeta function associated to the data $(\Gamma, \pi)$, defined for the case of $n = 3$, in the range of convergence $\mathrm{Re}(s) > 1$, as

$$Z_{\Gamma,\pi}(s) = \prod_{\{P_0\}\in\mathcal{R}} \prod_{j=1}^{\dim W} \prod_{\substack{l,k\geq 0 \\ c(P_0,j,l,k)=1}} \left(1 - \mathfrak{t}_j \lambda(P_0)^{-2k}\overline{\lambda(P_0)}^{-2l} N(P_0)^{-s-1}\right),$$

where

- $P_0$ ranges over $\mathcal{R}$, a *maximal reduced system* (in the sense of §5.4 of [6]) of $\Gamma$-conjugacy classes of primitive hyperbolic elements of $\Gamma$, and the torsion part of the centralizer of $P_0$ is generated by the elliptic element $E_{P_0}$.
- $\lambda(P_0)$ is the unique eigenvalue of $P_0$ satisfying $|\lambda(P_0)| > 1$.
- $N(P_0) := |\lambda(P_0)|^2$.
- $\mathfrak{t}_1, \ldots, \mathfrak{t}_n$ denote the eigenvalues of $\pi(P_0)$.
- $c(P_0, j, l, k) := \mathfrak{t}'_j \zeta(P_0)^{2l}\zeta(P_0)^{-2k}$, where the $\mathfrak{t}'_j$ are the eigenvalues of $\pi(E_{P_0})$ and $\zeta(P_0)$ is an eigenvalue of $E_{P_0}$ itself.

In this definition we follow Friedman ([8]), who is generalizing [6] and [22], and ultimately, [18].

The Selberg zeta function, originally introduced by Selberg in [18], and its relation to the spectral theory of $\Delta$ on $L^2$ is the object of numerous studies. In particular, the analytic continuation of the Selberg zeta function was studied in Chapter 7 of [22] and [8]. The main fact that will be vital to our argument is that, for Re$s > 1$,

$$\mathrm{d}\log Z_{\Gamma,\pi}(s)/\mathrm{d}s = \sum_{\{P\}_\Gamma} \frac{\mathrm{tr}_W \pi(P) \log NP_0}{m(P)|\lambda(P) - \lambda(P)^{-1}|^2} N(P)^{-s}, \tag{1}$$

where

- · $P$ in the summation ranges over a set of representatives of the conjugacy classes of all (not just primitive) hyperbolic elements in $\Gamma$.
- · $P_0$ is a primitive element of $\Gamma$ so that $P \in \langle P_0 \rangle$, the group generated by $P_0$.
- · $m(P)$ is the order of the torsion part of the element $E_P$, that is the order of the torsion part of the centralizer of $P$.

**The Problem.** In [21], [20], and Chapter 7 of [19], Venkov and Zograf developed the so-called Artin formalism for the Selberg zeta function. Namely, Venkov and Zograf proved the following inductivity result,

**Theorem 1** *Let $\Gamma' \subset \Gamma$ a subgroup of finite index of $\Gamma$, and $\psi$ a finite dimensional representation of $\Gamma'$. Then, for $\pi = Ind_{\Gamma'}^{\Gamma}\psi$, the following is true:*

$$Z_{\Gamma',\psi}(s) = Z_{\Gamma,\pi}(s).$$

Originally covering only the case of $n = 2$, that is finite volume surfaces $\Gamma\backslash\mathbf{H}$, Venkov's theorem has been recently extended to the case of $n = 3$ but not in complete generality. This paper extends Theorem 1 to the case of $n = 3$ in complete generality.

More precisely, in [9], Friedman proved Theorem 1 in the case of $n = 3$, but under the additional assumption that $\Gamma'$ is normal in $\Gamma$. We remove the normality assumption and prove Theorem 1 in the case of $n = 3$ for any pair $(\Gamma', \Gamma)$ such that $\Gamma'$ has finite index in $\Gamma$. In an obvious way, this result allows us to relate the Selberg zeta functions associated to *any pair* $(\Gamma, \Gamma')$ *of commensurable Kleinian groups* (the term *commensurable* meaning that $\Gamma \cap \Gamma'$ has finite index in both $\Gamma$ and $\Gamma'$).

Of equal or perhaps greater significance than the generalization, *per se*, of Theorem 1 in this paper is the method we use to obtain that result. The method differs from that used by our predecessors Venkov, Zograf, and Friedman, and instead follows a suggestion of [12]. What we mean by this is that we show the equality of the 'partial traces' of certain partially periodized kernels appearing in the trace formula, rather than the 'full trace', *i.e.*, the ('hyperbolic') term in the trace formula that gives the Selberg zeta function directly. For any $n$, once the 'partial traces' are shown to be equal, the result Theorem 1 follows directly as an 'image' of this equality under a further trace operation. We believe our method is valuable for two reasons. First, the equality of the partial traces that we establish generalizes to each of the pairs of terms entering into the respective 'pre-trace formulas' (*i.e.*, preliminary form of the trace formula before explicit calculation of the integrals). Thus our method exhibits a structural reason and broader context for the inductivity result of the Selberg zeta function. Second, comparing the partial traces of the hyperbolically periodized kernels, rather than their full traces, seems to make the proof more transparent, and apparently eliminates the need for invoking unnecessary assumptions (such as normality) to make the calculations more manageable.

The structure of the paper is as follows. In §2, we describe the setup and basic notions. In particular, we define the key notion of an *inductive pair of sections*, which allows us to predict when the trace of certain integral kernels will be equal, *before explicitly computing either trace*. In §3—a review of material found, *e.g.*, in [20]—we introduce the notion of Laplacian $\Delta$ appropriate to the function spaces we are working with and a map $T$ between the function spaces commuting with $\Delta$. The commutation relation implies that $T$ identifies the discrete spectra of the Laplacians of the respective spaces. In §4, we study the action of $T$ on the Eisenstein series and Eisenstein kernels, and hence on the continuous spectrum. In §5, we review the main facts from the theory of the Selberg trace formula, especially the convergence of the regularized integral 'over the diagonal', in so far as these are required in order to show that each of the (regularized) terms, in the case of a non-compact quotient, actually have convergent traces. In §6, we show that each of the 'pairs' of corresponding sections, whose trace is to be computed in the trace formulas for $\Gamma$, $\Gamma'$, indeed form an inductive pair of sections (in the sense defined in §2), and we use the previously mentioned results to prove that each of the pairs of corresponding terms in the respective trace formulas are equal. The remainder of the section is devoted to three closely related applications. When we take a specific kernel, called the Green's kernel, and focus on the special case of $\Omega$-hyperbolic elements, we obtain a short proof of the Artin formalism, Theorem 1, as explained in [4]. When we take a general kernel which satisfies certain analytic conditions of Selberg, we obtain the equality of each of the pairs of corresponding 'orbital integrals' in the trace formulas of $\Gamma$, $\Gamma'$. Finally, when we take a certain kernel of this type, called the heat kernel, we obtain from the hyperbolic contribution to the trace formula a function whose integral transform is, roughly speaking, the Selberg zeta function, The choice of the heat kernel as 'test kernel' allows us to unify the two previous applications under one perspective. The remaining two sections, §7 and §8, are devoted respectively to a 'computational' example of the newly generalized Theorem 1 having particular interest, and to a brief discussion on the prospects for extensions to higher dimensions, this discussion being in some ways motivated by the example of §7.

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## 2 Basic Notions: measures, discrete subgroups, representations, and vector bundles.

**Setup.** From now until §7, all of the following notations and notions will be fixed.

- $G = \mathrm{SL}(2, \mathbf{C})$, $K = \mathrm{SU}(2)$.
- $\mathbf{H} = \mathbf{H}^3 = \{z = x + \mathbf{j}y \mid x = x_1 + x_2\mathbf{i} \in \mathbf{C}, y > 0\}$, acted upon by $G$ by fractional linear transformations.
- $\mathrm{d}z^2$ is the fixed $G$-invariant metric and $\mu$ is the fixed $G$-invariant measure on $\mathbf{H}$ given by
$$\mathrm{d}z^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}, \quad \mathrm{d}\mu = \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y}{y^3}.$$
- $\Gamma$, $\Gamma'$ are lattices, *i.e.* Kleinian groups of finite covolume, each containing $\{\pm I_2\}$.
- $\Gamma'$ is a subgroup of $\Gamma$, and the index $[\Gamma : \Gamma'] = n$.
- Depending on the context, $\mu$ may denote the (quotient) hyperbolic measure on either $\Gamma\backslash\mathbf{H}$ or $\Gamma'\backslash\mathbf{H}$, so that, in particular,
$$\mu(\Gamma'\backslash\mathbf{H}) = n\mu(\Gamma\backslash\mathbf{H}).$$
- $\mathbf{P}^1(\mathbf{C}) = \partial\mathbf{H}$ is the boundary of $\mathbf{H}$, with
$$\overline{\mathbf{H}} = \mathbf{H} \cup \partial\mathbf{H}.$$
- For $a \in \overline{\mathbf{H}}$, $\Gamma_a$ is the stabilizer of $a$ in $\Gamma$, and for $a \in \mathbf{P}^1(\mathbf{C})$, $\Gamma_a^U$ is the group of unipotent elements in $\Gamma_a$.

For purposes of many definitions $\Gamma$ and $\Gamma'$ are on an equal footing, and we will not take the time to restate definitions for $\Gamma'$ (*e.g.*, of $\Gamma'_a$) when it is obvious how to do so.

**Cusps and Coset representatives.** We begin with

**Definition 2** *A point $a \in \mathbf{P}^1(\mathbf{C})$ is called a **cusp of $\boldsymbol{\Gamma}$** precisely when $\Gamma_a^U$ contains a free abelian group of rank 2. The set of cusps of $\Gamma$ will be denoted* **Cusp(Γ)**. *A collection of $\Gamma$-inequivalent cusps will be denoted by $\boldsymbol{C(\Gamma)}$. We denote the (finite) cardinality $|C(\Gamma)|$ by $\boldsymbol{h(\Gamma)}$.*

By Proposition 2.3.7 of [6], a consequence of 'Shimizu's Lemma', Theorem 2.3.1 (of [6]), the condition that $\Gamma_a^U$ contains a free abelian group of rank 2 is equivalent to the condition $\Gamma_a^U \neq \{I_2\}$. The condition $|C(\Gamma)| < \infty$ is proved in Proposition 2.3.8 of [6].

**Lemma 3** *In the above setup,*

*(a)* *We have* $\mathrm{Cusps}(\Gamma') \subseteq \mathrm{Cusps}(\Gamma)$.
*(b)* *For any choice of $C(\Gamma) \subseteq \mathrm{Cusps}(\Gamma)$, or any choice of $C(\Gamma') \subseteq \mathrm{Cusps}(\Gamma')$, we can select the other representative set so that*

$$C(\Gamma) \subseteq C(\Gamma').$$

*Proof.* For (a), note that for any $a \in \mathbf{P}^1(\mathbf{C})$, the index of ${\Gamma'}_a^U$ in $\Gamma_a^U$ is at most $n < \infty$. Therefore, when $\Gamma_a^U$ is free abelian of rank 2, ${\Gamma'}_a^U$ must be free abelian of rank two as well. Part (b) is clear from the fact that a $\Gamma$-orbit in $\mathrm{Cusps}(\Gamma)$ splits into finitely many $\Gamma'$-orbits (at most $n$ such). □

We will denote the subset of $C(\Gamma')$ consisting of cusps which are $\Gamma$-equivalent to a given $a \in C(\Gamma)$ by $C_a(\Gamma')$. Clearly, we have the partition

$$C(\Gamma') = \dot{\bigcup_{a \in C(\Gamma)}} C_a(\Gamma').$$

Our convention is always to take $a$ as an element of $C_a(\Gamma')$, and to put $a$ first in any ordering of the elements of $C_a(\Gamma')$. We set

$$h_a(\Gamma') = |C_a(\Gamma')|.$$

In homage to the situation of Galois theory, with which a strong analogy will emerge in the course of the paper, we will usually denote the situation $b \in C_a(\Gamma')$ by writing $\boldsymbol{b|a}$. In addition we will sometimes have occasion to apply this notation to the cusps themselves. Naturally, when applied to cusps $b|a$ rather than inequivalent representatives of the cusps,

$b|a$ means that $b \in \mathrm{Cusps}(\Gamma')$, $a \in \mathrm{Cusps}(\Gamma)$ and $b$ is $\Gamma$-equivalent to $a$.

From now on, for $b|a$, we fix $\sigma_{ab} \in \Gamma$ such that

$$\sigma_{ab} a = b.$$

**Definition 4** *We have a natural concept of the* ***width*** $\boldsymbol{n_b}$ ***of*** $\boldsymbol{b} \in$ **Cusps**$(\boldsymbol{\Gamma'})$*, given by setting*

$$n_b = [\Gamma_b : \Gamma'_b].$$

*Clearly $n_b$ depends only on the $\Gamma'$-equivalence class of $b$, so we may speak unambiguously of $n_b$ for the class of $b \in C(\Gamma')$.*

The next statement, Proposition 5, says that the coset representatives for $\Gamma' \backslash \Gamma$, can be chosen to be "consistent" in a certain precise sense with a choice of cusp $a \in \mathrm{Cusp}(\Gamma)$ and a choice of representatives of the $\Gamma$-equivalent cusps $C_a(\Gamma')$.

**Proposition 5** *Let $a \in C(\Gamma)$. For $b|a$, let*

$$\{\beta_{bj} \mid 1 \leq j \leq n_b\}$$

*be any chosen representatives for the cosets of $\Gamma'_b \backslash \Gamma_b$. Then we have the disjoint decomposition*

$$\Gamma = \bigcup_{b|a} \bigcup_{j=1}^{n_b} \Gamma' \beta_{bj} \sigma_{ab}. \tag{2}$$

Where doing so improves readability, we will write the coset representatives $\beta_{bj}\sigma_{ab}$ in (2) more simply as $\{\alpha_i\}$, with $i$ ranging from 1 to $n$.

**Corollary 6** *For $a \in C(\Gamma)$, we have*

$$\sum_{b|a} n_b = n.$$

Proposition 5 is a direct application of the more general Proposition 29, whose proof is not difficult but somewhat tedious. We place the proof of both propositions in the Appendix so as not interrupt the main flow of the argument.

**Unitary Representations.** Let $(\psi, V)$ denote a finite-dimensional (unitary) representation of $\Gamma'$. Let $\langle \cdot, \cdot \rangle$ denote the inner product of $V$ giving rise to the norm $|\cdot|_V$. The assumption that $(\pi, V)$ is unitary means that for every $\gamma' \in \Gamma'$, the operator $\psi(\gamma')$ preserves $\langle \cdot, \cdot \rangle$ and $| \cdot |$.

There is an induced representation of $(\mathrm{ind}_{\Gamma'}^{\Gamma} \psi, W)$ of $\psi$ on the space of functions.

$$W = \{\phi : \Gamma \to V \mid \phi(\gamma g) = \psi(\gamma)\phi(g) \text{ for all } \gamma \in \Gamma',\ g \in \Gamma\} \tag{3}$$

transforming by $\psi$ under $\Gamma'$ on the left. The larger group $\Gamma$ acts by right-translation on $W$. Henceforth, whenever $(\psi, V)$ is our fixed representation of $\Gamma'$, we will use $(\pi, W)$ to denote $(\mathrm{ind}_{\Gamma'}^{\Gamma}, W)$. Now consider the representation $(\pi, V^n)$ of $\Gamma$ on $V^n$ defined by setting

$$\pi(\gamma) \in \mathrm{End}(V^n) \text{ with coefficient } \pi(\gamma)_{ij} = \tilde{\psi}(\alpha_i \gamma \alpha_j^{-1}). \tag{4}$$

Here, we use the notation

$$\tilde{\psi}(\gamma) = \begin{cases} \psi(\gamma) & \text{if } \gamma \in \Gamma' \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

It is not difficult to see that $\mathrm{dim}W = n\,\mathrm{dim}V$ and, indeed, there is a $\Gamma$-intertwining isomorphism

$$L_a : V^n \to W,\ L_a(\oplus_i v_i) = f,\ \text{where } f(\alpha_i) = v_i, \text{ for } i = 1, \ldots n. \tag{6}$$

Thus $(\pi, V^n)$ and $(\pi, W)$ are indeed equivalent realizations of the same representation of $\Gamma$. From (6), one sees that

$$L_a = \pi(\sigma_{ab}^{-1}) \circ L_b.$$

The inner product $\langle \cdot, \cdot \rangle_V$ on $V$ induces an inner product on $W$ by

$$\langle \phi_1, \phi_2 \rangle_W = \sum_{i=1}^{n} \langle \phi_1(\alpha_i), \phi_2(\alpha_i) \rangle_V,$$

and an induced norm $||\cdot||_W$ satisfying

$$||\phi||_W^2 = \sum_{i=1}^{n} ||\phi(\alpha_i)||_V^2.$$

Furnishing $V^n$ with the 'direct-sum' metric, meaning that

$$||\oplus v_i||_{V^n}^2 := \sum_i ||v_i||_V^2,$$

we see that

$$\text{the vector isomorphism } L_a \text{ is an isometry from } (V^n, |\cdot|_{V^n}) \text{ to } (W, |\cdot|_W). \tag{7}$$

It is easy to deduce from these definitions that $\langle \cdot, \cdot \rangle_W$ is independent of $a$, *i.e.*, of the choice of the $\{\alpha_i\}$. It is readily verified that the operators $\pi(\gamma)$ for $\gamma \in \Gamma$ preserve $\langle \cdot, \cdot \rangle_W$, and therefore $(\pi, W)$ is also a unitary representation.

**Vector Bundles.** We systematically generalize the more familiar picture involving $\Gamma'$ automorphic functions to $V$-valued functions transforming under $\Gamma'$ according to $\psi$. In order to do so, it will be convenient to use the language of vector bundles and mappings between spaces of sections, following [2], for example. Begin with the trivial vector bundle based on $\mathbf{H}^3$, with fiber $V$:

$$\mathbf{H}^3 \times V.$$

Consider the equivalence relation

$$(\gamma z, v) \overset{\Gamma'}{\sim} (z, \psi(\gamma^{-1})v), \text{ for all } \gamma \in \Gamma'. \tag{8}$$

The quotient of $\mathbf{H}^3 \times V$ by this equivalence relation is denoted by

$$\mathcal{H}_{\psi,V} := \mathbf{H}^3 \times V/(\sim, \Gamma').$$

Similarly, we define

$$\mathcal{H}_{\pi,W} := \mathbf{H}^3 \times W/(\sim, \Gamma),$$

and $\mathcal{H}_{\pi,V^n}$. We denote the continuous sections of $\mathcal{H}_{\psi,V}$ of compact support by $C_0(\mathcal{H}_{\psi,V})$, the space of smooth sections by $C_0(\mathcal{H}_{\psi,V})$. Similarly, for the other bundles that arise in this discussion.

The metric on $V$ together with the hyperbolic measure on $\mathbf{H}^3$ induce an inner product and a norm $||\cdot||_V$ on $C_0(\mathcal{H}_{\psi,W})$. Specifically, if $f \in C_0(\mathcal{H}_{\psi,V})$, then

$$|f|_V^2 := \int_{\Gamma\backslash\mathbf{H}^3} ||f(z)||_V^2 \, \mathrm{d}\mu(z).$$

We define

$$L^2(\Gamma', \psi, V) = \text{completion of } C_0(\mathcal{H}_{\psi,V}) \text{ in } ||\cdot||_V.$$

Similarly, we obtain $L^2(\Gamma, \pi, W)$ , resp. $L^2(\Gamma, \pi, V^n)$, by completing $C_0(\mathcal{H}_{\pi,W})$, $C_0(\mathcal{H}_\psi, V)$ in the induced metrics. We often abbreviate $L^2(\Gamma', \psi, V)$ by $L^2(\Gamma', \psi)$ and $L^2(\Gamma, \pi, W)$ by $L^2(\Gamma, \pi)$. Similarly, we drop the $w$ in $\mathcal{H}_{\pi,W}$ and write simply $\mathcal{H}_\pi$.

We now define a number of operations between the spaces of sections of the various vector bundles, all ostensibly depending on a choice of coset representative $\{\alpha_i\}$ for $\Gamma'$ in $\Gamma$, associated to a cusp $a \in C(\Gamma)$. First, we have

$$T_{a,V} : C_0(\mathcal{H}_\psi) \to C_0(\mathcal{H}_{\pi,V^n}), \text{ defined by } f(\cdot) \mapsto \oplus_i f(\alpha_i \cdot).$$

Naturally, $T_{a,V}$ extends to the $L^2$-sections. Further, the isometry $L_a$ of the fiber $V^n$ onto $W$ yields an isometry of the section spaces, also denoted by $L_a$,

$$L_a : C_0(\mathcal{H}_{\pi,V^n}) \to C_0(\mathcal{H}_\pi),$$

We also use $L_a$ to denote the extension of the isometry to $L^2$-sections. Next we define $T_a$ as the composition

$$L^2(\Gamma', \psi) \xrightarrow{T_{a,V}} L^2(\Gamma, \pi, V^n) \xrightarrow{L_a} L^2(\Gamma, \pi),$$

so that $T_a = L_a \circ T_{a,V}$.

**Proposition 7** *The map $T_a : L^2(\Gamma', \psi) \to L^2(\Gamma, \pi)$ has the following properties.*

*(a) $T_a$ does not depend on the choice of cusp $a$. Therefore, we will denote $T_a$ henceforth by $T$.*
*(b) $T$ is an isometry of $L^2$-spaces, meaning that,*

$$\int_{\Gamma'\backslash\mathbf{H}^3} ||f(z)||_V^2 \mathrm{d}z = \int_{\Gamma\backslash\mathbf{H}^3} ||Tf(z)||_W^2 \mathrm{d}z, \text{ for all } f \in L^2(\Gamma', \psi).$$

*Proof.* (a) Assume that $f$ belongs to the dense subspace $C_0(\mathcal{H}_\psi)$. Then $T_a f \in C_0(\mathcal{H}_\pi)$. Note that $(T_a f)(z) \in W$, the representation space of $\pi$, the function $(T_a f)(z) : \Gamma \to W$ is determined by its value on the $\alpha_i$. We compute, using (6), that

$$(T_a f)(z)(\alpha_i) = (L_a(T_a f))(z))(\alpha_i) = L_a(\oplus_j f(\alpha_j z))(\alpha_i) = f(\alpha_i z),$$

which has no dependence on $a$. (b) We calculate,

$$\int_{\Gamma'\backslash\mathbf{H}} ||f(z)||_V^2 \,\mathrm{d}z = \int_{\Gamma\backslash\mathbf{H}} \sum_i ||f(\alpha_i z)||_V^2 \,\mathrm{d}z = \int_{\Gamma\backslash\mathbf{H}} || \oplus_i f(\alpha_i z)||_{V^n}^2 \,\mathrm{d}z =$$
$$= \int_{\Gamma\backslash\mathbf{H}} ||L_a T_{a,V} f(z)||_W^2 \,\mathrm{d}z,$$

where we have used (7) in the next-to-last equality. □

As we will see the map $T$ is central to all that follows, primarily because of its additional property of intertwining the Laplacians (Proposition 10).

We use certain trivial vector bundles to treat systematically certain functions which are not automorphic but satisfy a weaker condition which is more natural to consider in this context. The trivial vector bundles are always indicated by $\mathcal{E}$, in contrast to the non-trivial vector bundles, always indicated by $\mathcal{H}$. According to our notation convention, the trivial bundle $E_*$ always has fiber equal to the endomorphism ring of the space $*$. Namely,

- $\mathcal{E}_V = \mathbf{H}^3 \times \mathrm{End}V$;
- $\mathcal{E}_W = \mathbf{H}^3 \times \mathrm{End}W$;
- $\mathcal{E}_{V^n} = \mathbf{H}^3 \times \mathrm{End}V^n$.

We define certain operations among the spaces of sections of the $\mathcal{E}$-bundles. The first operator, $\mathrm{tr}_V^W$, is defined fiber-wise using $L_a$ (cf. the definition of $T_a$). Namely, the trace operator

$$\mathrm{tr}_V^{V^n} : \mathrm{End}(V^n) \to \mathrm{End}(V),$$

gives a natural 'trace' operation from $C^\infty(\mathcal{E}_{V^n})$ to $C^\infty(\mathcal{E}_V)$. By composition with $L_a^{-1}$, we obtain

$$\mathrm{tr}_V^W : C^\infty(\mathcal{E}_W) \xrightarrow{L_a^{-1}} C^\infty(\mathcal{E}_{V^n}) \xrightarrow{\mathrm{tr}_V^{V^n}} C^\infty(\mathcal{E}_V).$$

Along the same lines, we have the natural fiberwise maps,

- $\mathrm{tr}_V : C^\infty(\mathcal{E}_V) \to C^\infty(\mathbf{H})$;
- $\mathrm{tr}_W : C^\infty(\mathcal{E}_W) \to C^\infty(\mathbf{H})$.

Given a set of coset representatives $\{\alpha_i\}$ for $\Gamma'$ in $\Gamma$, as above, we define

$$\mathrm{tr}_{\{\alpha_i\}} \text{ on } C^\infty(\mathcal{E}_V) \text{ by } \left(\mathrm{tr}_{\{\alpha_i\}} f\right)(z) = \sum_{i=1}^{n} f(\alpha_i z)$$

Unlike $\mathrm{tr}_V^W$, the operator $\mathrm{tr}_{\{\alpha_i\}}$ depends on $\{\alpha_i\}$. However, $\mathrm{tr}_{\{\alpha_i\}}$ becomes independent of the choice of $\{\alpha_i\}$ *when restricted to the the subspace of sections whose trace is invariant.* More precisely, define the "trace-invariant" subspaces

- $C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'} = \{f \in C^\infty(\mathcal{E}_V) \mid \mathrm{tr}_V(f(\cdot) - f(\gamma\,\cdot)) \equiv 0 \text{ for all } \gamma \in \Gamma'\}$;
- $C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma} = \{f \in C^\infty(\mathcal{E}_V) \mid \mathrm{tr}_V(f(\cdot) - f(\gamma\,\cdot)) \equiv 0 \text{ for all } \gamma \in \Gamma\}$;
- $C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma} = \{f \in C^\infty(\mathcal{E}_W) \mid \mathrm{tr}_W(f(\cdot) - f(\gamma\,\cdot)) \equiv 0 \text{ for all } \gamma \in \Gamma\}$.

By definition, we have

- $\mathrm{tr}_V : C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'} \to C^\infty(\Gamma'\backslash\mathbf{H})$;
- $\mathrm{tr}_W : C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma} \to C^\infty(\Gamma\backslash\mathbf{H})$.

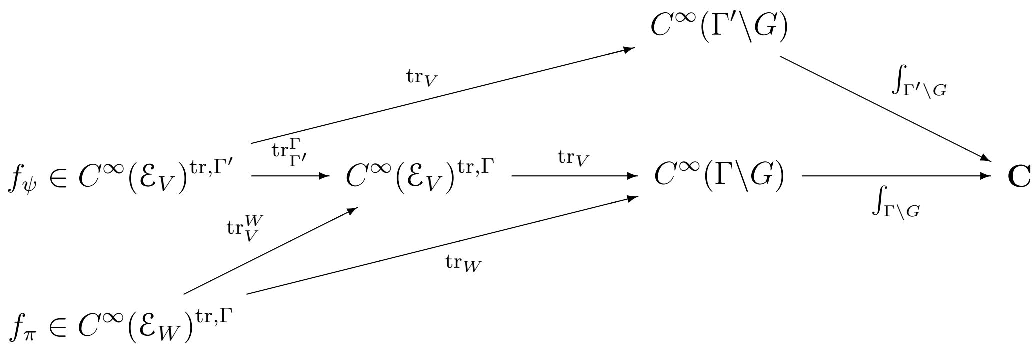
It is not difficult to verify that

- $\mathrm{tr}_V^W : C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma} \to C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma}$,
- $\mathrm{tr}_{\{\alpha_i\}} : C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'} \to C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma}$.

Further, one can easily verify that, *when restricted to $C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'}$, the operator $\mathrm{tr}_{\{\alpha_i\}}$ becomes independent of the choice $\{\alpha_i\}$ of coset representatives.* Further, the image of the restricted operator is easily seen to lie in $C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma}$. So, we may denote the restriction of $\mathrm{tr}_{\{\alpha_i\}}$ by

$$\mathrm{tr}_{\Gamma'}^{\Gamma} :\ C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'} \to C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma}.$$

We can put all these maps into a diagram, and it is elementary to verify that the resulting diagram is commutative.




When $f_\psi \in C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'}$, $f_\pi \in C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma}$ as in the diagram, we set

$$F_\psi = \mathrm{tr}_V f_\psi \ \text{ and } F_\pi = \mathrm{tr}_W f_\pi, \tag{9}$$

so that $F_\psi \in C^\infty(\Gamma'\backslash G)$ and $F_\pi \in C^\infty(\Gamma\backslash G)$. Provided, further, that the integrals converge, we define

$$I_\psi = \int_{\Gamma'\backslash G} F_\psi, \ \text{ and } I_\pi = \int_{\Gamma\backslash G} F_\pi. \tag{10}$$

**Definition 8** *If $f_\psi$ and $f_\pi$, as in the diagram, satisfy*

$$\mathrm{tr}^{\Gamma}_{\Gamma'} f_\psi = \mathrm{tr}^W_V f_\pi, \tag{11}$$

*then we call $(f_\psi, f_\pi)$ an **inductive pair of sections**.*

We read off from the commutative diagram that

$(f_\psi, f_\pi)$ is an inductive pair of sections implies that $\mathrm{tr}^{\Gamma}_{\Gamma'} F_\psi = F_\pi$, and $I_\psi = I_\pi$.

## 3 The Laplacians and the correspondence of their spectra

On $C^\infty(\mathcal{H}_\psi)$, the Laplacian can be defined in the "naive way" as a second-order differential operator,

$$\Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial}{\partial y}.$$

There is a well-known a theory extending $\Delta$ to an *unbounded, essentially self-adjoint operator* $\boldsymbol{\Delta(\Gamma', \psi)}$ *on* $L^2(\Gamma', \psi)$. We will not repeat this development here, but refer the reader to [16], Chapter VIII for the general theory of such extensions of operators, and Chapter IV of [6] (scalar, $\psi = 1$, case), or §3.2

of [7], for specific application to the case at hand. One fact that simplifies our discussion, is that within $L^2(\Gamma', \psi)$, the eigenfunctions of $\Delta(\Gamma'; \psi)$ are smooth. See *e.g.* [6] for the proof in the 'scalar' case, where $\psi$ is the trivial one-dimensional representation. The 'vector' case, where $\psi$ is a general finite dimensional unitary representation, is similar.

Since $\Delta(\Gamma', \psi)$ is a positive operator, its eigenvalues in $L^2(\Gamma, \psi)$ are positive, and can be arranged in an increasing sequence

$$0 \leq \lambda_1(\Gamma', \psi) \leq \lambda_2(\Gamma', \psi) \leq \cdots ,$$

counted according to multiplicity. Let $\phi_j \in L^2(\Gamma', \psi)$ be the normalized eigenfunction of $\Delta(\Gamma'; \psi)$ with eigenvalue $\lambda_j(\Gamma', \psi)$.

**Definition 9** *We refer to the ordered set of eigenvalues* $\{\lambda_j(\Gamma', \psi)\}$ *as the* ***discrete spectrum of*** $\boldsymbol{\Delta(\Gamma', \psi)}$***.*** *We also use the notation*

$$L^2_{\text{disc}}(\Gamma', \psi) := \text{span}_{\mathbf{C}}(\{\phi_j\}).$$

Note that, at a minimum, $L^2_{\text{disc}}(\Gamma', 1)$ contains the constant function of norm 1, since a constant function an eigenfunction of $\Delta(\Gamma'; 1)$ with corresponding eigenvalue 0. It is not easy to determine how large $L^2_{\text{disc}}(\Gamma', 1)$ is, nor even if it is infinite dimensional, and the answer to that question is known to depend on $\Gamma'$. Selberg and others have studied that question in detail, and there is an asymptotic result known as the *Weyl Law for* $\Gamma'$, but we will not need such results here.

We have a similar sequence of eigenvalues for $\Delta(\Gamma, \pi)$. We denote by $\Phi_j \in L^2(\Gamma, \pi)$ the eigenfunction of $\Delta(\Gamma, \pi)$ with eigenvalue $\lambda_j(\Gamma, \pi)$, and by $L^2_{\text{disc}}(\Gamma, \pi)$ the span of the $\Phi_j$. The discrete spectra of the two Laplacians correspond under the mapping $T$ of §2. More precisely,

**Proposition 10** *We have*

*(a)* *$T$ commutes with $\Delta$, or more precisely, the equality*

$$\Delta(\Gamma, \pi) = T \circ \Delta(\Gamma', \psi) \circ T^{-1},$$

*is valid on the appropriate domain in $L^2(\Gamma, \pi)$.*

*(b)* *As a consequence, $T\phi_j = \Phi_j$, and $\lambda_j(\Gamma', \psi) = \lambda_j(\Gamma, \pi)$. So, $T$ induces an isometry*

$$T : L^2_{\text{disc}}(\Gamma', \psi) \to L^2_{\text{disc}}(\Gamma, \pi),$$

*thus identifying eigenspaces of the Laplacians with the same eigenvalue.*

*(c)* *The mapping $T$, extended in the natural way to the space of smooth sections, defines a linear isomorphism of the corresponding automorphic forms of*

*weight $s(2-s)$, denoted by*

$$T : \mathcal{A}(\Gamma', \psi, s) \cong \mathcal{A}(\Gamma, \pi, s).$$

*Proof.* Following [20], p. 438, first consider the restriction of $\Delta(\Gamma', \psi)$ to the smooth sections $C^\infty(\mathcal{H}_\psi)$, so that the operator is given by the 'naive' second-order differential operator. The definition of $\Delta(\Gamma', \psi)$ says that $\Delta(\Gamma', \psi)$ acts on a section in $C^\infty(\mathcal{H}_\psi)$ as the scalar Laplacian, applies to each coordinate function associated the section. Combine this observation with the characterization of the 'naive' Laplacian as the unique second-order differential operator commuting with all translations by group elements. From the fact that $T$ is defined as a direct sum of translations by group elements, we deduce $T \circ \Delta(\Gamma', \psi) \circ T^{-1}$ must act as $\Delta(\Gamma, \pi)$ on the smooth sections. By the uniqueness of the self-adjoint extension to (the appropriate dense subspace of $L^2(\Gamma, \pi)$), we conclude the proof of part (a).

Part (b) follows immediately from Part (a), and the definitions preceding Proposition 10.

Part (c) is proved in §4, below. □

We sometimes sum up part (b) by saying that $T$ identifies the *discrete spectra* of the Laplacians $\Delta(\Gamma', \psi)$ and $\Delta(\Gamma, \pi)$ and part (c) by saying that $T$ identifies the *continuous spectra* of the Laplacians.

## 4 Eisenstein Series

We must define the vector-valued Eisenstein series that are needed to write spectral expansions of the point-pair invariants, and hence the Eisenstein kernels, occurring in the Selberg trace formula for $\mathcal{H}_\psi$. We establish some conventions, building on the notation introduced in §2.

- When $a$ denotes an element of Cusps($\Gamma$) (respectively $C(\Gamma)$) $b$ denotes an element of Cusps($\Gamma'$) (respectively $C(\Gamma')$) such that $b|a$.
- $V_b := \{v \in V \mid \psi(\gamma)v = v, \text{ for all } \gamma \in \Gamma'_b\}$ is called the **singular space of $\boldsymbol{b}$ with respect to $\boldsymbol{\psi}$**.
- For $a|b$ and $\pi$ a representation of $\Gamma$, we set

  $$[\pi : 1] := \dim_{\mathbf{C}} \mathrm{Hom}_\Gamma(1, \pi),$$

  and similarly for representations of $\Gamma'$. (The index notation furthers the analogy with Galois theory introduced by means of the notation $a|b$.)
- $W_a := \{w \in W \mid \pi(\gamma)w = w, \text{ for all } \gamma \in \Gamma_a\}$.

- $k_\psi(b) = \dim V_b$, the **degree of singularity of $b$ with respect to $\psi$**.
- $k_\pi(a) = \dim W_a$.
- $\kappa(\psi) = \sum_{b\in C(\Gamma')} k_b(\psi)$ (respectively $\kappa(\pi) = \sum_{a\in C(\Gamma)} k_a(\pi)$), the **full degree of singularity of $\psi$ (respectively, of $\pi$)**.

Note that for $\gamma \in \Gamma'$, $V_{\gamma b} = \psi(\gamma)V_b$, so that the dimensions $k_\psi(b)$ do not depend on the choice of cuspidal representatives $C(\Gamma')$ among Cusps$(\Gamma')$. A similar comment applies for the degree of singularity $k_\pi(a)$.

**Degree of Singularity.** Recall, from §2, the following equalities

$$n = \sum_{b|a} n_b \quad \text{and} \quad h_a(\Gamma') = \sum_{b|a} 1.$$

Further, the entries of a vector in $V^n$ may be grouped into $h_a(\Gamma')$ blocks, each block consisting of $n_b$ contiguous entries. We denote by $\imath_b$ the obvious injection $V^{n_b} \hookrightarrow V^n$ into the $b^{\text{th}}$ block of entries, with 0's in the remaining $h_a(\Gamma') - 1$ blocks. Composing with the isomorphism $L_a$ we have a metric-preseving injection of $V^{n_b}$ into $W$:

$$V^{n_b} \overset{\imath_b}{\hookrightarrow} V^n \overset{L_a}{\longrightarrow} W.$$

Since the image of the $L_a \circ i_b$ (for $b|a$) clearly span $W$, we have

$$W = \bigoplus_{b|a} (L_a \circ \imath_b)V^{n_b}. \tag{12}$$

Further, the summands in (12) can be characterized succinctly. By the definition (3) of $W$ as the module for the representation $\pi$, an element $\mathbf{w} \in W$ is determined by its restriction to the set of coset representatives $\{\beta_{cj}\sigma_{ac}\}$, $c|a$ (see (2)), $c|a$ (see 2). Then we have

$$\mathbf{w} \in (L_a \circ \imath_b)V^{n_b} \text{ if and only if } \mathbf{w}(\beta_{cj}\sigma_{ac}) \text{ vanishes for } c \neq b. \tag{13}$$

For $a|b$, $\mathbf{f} \in V_b$, we define $\mathbf{f}^a$ as the image

$$\mathbf{f}^a = L_a \circ \imath_b(n_b^{-1/2}(\mathbf{f}, \ldots, \mathbf{f})).$$

It is not difficult to see that we have the relationship

$$\mathbf{f}^b = \pi(\sigma_{ab})\mathbf{f}^a \text{ for } b|a. \tag{14}$$

**Proposition 11** *For $\gamma \in \Gamma_a$, $\pi(\gamma)$ is 'block diagonal', in the sense that*

*(a) $\pi(\gamma)$ for $\gamma \in \Gamma_a$ preserves each subspace $(L_a \circ \imath_b)V^{n_b}$ of $W$ in the direct sum decomposition* (12). *Part (a) has the following consequences*

*(b) For $\mathbf{f} \in V_b$, $\mathbf{f}^a \in W_a$.*

(c) *The degree of singularity* $k_\pi(a) = \dim W_a$ *equals* $\sum_{b|a} k_\psi(b)$*, so that we have equality of the total degrees of singularities,* $\kappa(\pi) = \kappa(\psi)$.

(d) *More precisely, for* $\{\mathbf{f}_j\}_{j=1}^{n_b}$ *an (orthonormal) basis of* $V_b$*, the set*

$$\{\mathbf{f}_j^a \mid b|a,\ 1 \le j \le n_b\}$$

*is an (orthonormal) basis for* $W_a$.

*Proof.* Let $\gamma \in \Gamma_a$. As an element of the function space $W$, $\pi(\gamma)\mathbf{f}$ is determined by its value on the coset representatives $\beta_{cj}\sigma_{ac}$. Let $\mathbf{f} \in (L_a \circ \imath_b)V^{n_b}$. Then, $\mathbf{f}(\Gamma'\beta_{ck}\sigma_{ac}) = 0$ if $c \neq b$, for all $\gamma' \in \Gamma'$, by the above characterization (13). We have further, because $f \in W$,

$$\pi(\gamma)\mathbf{f}(\beta_{cj}\sigma_{ac}) = \mathbf{f}(\beta_{cj}\sigma_{ac}\gamma)$$

However, note that, since $\gamma \in \Gamma_a$,

$$(\beta_{cj}\sigma_{ac}\gamma)a = c.$$

This shows that

$$\beta_{cj}\sigma_{ac}\gamma \in \Gamma'\beta_{ck}\sigma_{ac},$$

and therefore, if $c \neq b$,

$$\pi(\gamma)\mathbf{f}(\beta_{cj}\sigma_{ac}) = \mathbf{f}(\beta_{cj}\sigma_{ac}\gamma) = 0,$$

by the above characterization of $\mathbf{f}$. By the characterization of $(L_a \circ \imath_b)V^{n_b}$ above we see that $\pi(\gamma)\mathbf{f} \in (L_a \circ \imath_b)V^{n_b}$. This completes the proof of (a).

For (b), note that in order to check that $\pi(\gamma)\mathbf{f}^a = \mathbf{f}^a$, when $\gamma \in \Gamma_a$, all that remains, in light of what we have shown so far, is to show that

$$\pi(\gamma)\mathbf{f}^a(\beta_{bj}\sigma_{ab}) = \mathbf{f},\ \text{for } 1 \le j \le n_b.$$

Note that since $\gamma \in \Gamma_a$, we have $\gamma = \beta_{ak}\gamma'$, for some $1 \le k \le n_a$ and some $\gamma' \in \Gamma$. Further,

$$\beta_{bj}\sigma_{ab}\beta_{ak}a = b, \text{ so that } \beta_{bj}\sigma_{ab}\beta_{ak} \in \beta_{bl}\sigma_{ab}\Gamma',\ \text{for some } 1 \le l \le n_b.$$

Set

$$\gamma'' = (\beta_{bl}\sigma_{ab})^{-1}\beta_{bj}\sigma_{ab}\beta_{ak} \in \Gamma'.$$

We know $\mathbf{f}^a(\beta_{bj}\sigma_{ab}) = \mathbf{f}$, by the definition of $\mathbf{f}^a$. But

$$(\pi(\gamma)\mathbf{f}^a)(\beta_{bj}\sigma_{ab}) = \mathbf{f}^a(\beta_{bj}\sigma_{ab}\gamma) = \mathbf{f}^a(\beta_{bl}\sigma_{ab}\gamma'') = \psi(\gamma'')\mathbf{f}^a(\beta_{bl}\sigma_{ab}) = \psi(\gamma'')\mathbf{f},$$

since $\gamma'' \in \Gamma'$. But also note that $\gamma'' \in \Gamma'_b$, so that $\psi(\gamma'')\mathbf{f} = \mathbf{f}$. We have therefore, shown that $\pi(\gamma)\mathbf{f}^a(\beta_{bj}\sigma_{ab}) = \mathbf{f}$, as was required.

For (c), that $\pi(\gamma)$ is block-diagonal for $\gamma \in \Gamma_a$ implies that

$$\pi|_{\Gamma_a} = \oplus_{b|a} \mathrm{Ind}_{\Gamma_b}^{\Gamma_a}(\psi|_{\Gamma'_b}).$$

We then have by Frobenius reciprocity

$$k_a(\pi) = [\pi|_{\Gamma_a} : 1] = \sum_{b|a} [\mathrm{Ind}_{\Gamma'_b}^{\Gamma_b}(\psi|_{\Gamma'_b}) : 1] = \sum_{b|a} [\psi|_{\Gamma'_b} : 1] = \sum_{b|a} k_b(\psi).$$

In view of part (c), part (d) follows by observing that the $\{\mathbf{f}_j^b\}$ for $b|a$ and $1 \leq j \leq n_b$, form a set of linearly independent vectors of the cardinality equal to $\dim W_a$. $\square$

**Scaling Matrices.** The other new element we need to introduce before the definition the vector Eisenstein series associated to the cusp $a$ is the **scaling matrix** $S_a$. Our definition of the scaling matrix, which differs from that found in previous literature on the subject, turns out to eliminate certain 'extra factors' that pop up in the Eisenstein series correspondence, as found in previous treatments. These extra factors in our view somewhat obscure the essential matter.

**Definition 12** *Fix $a \in C(\Gamma)$. Associate an element $S_a \in \mathrm{PSL}_2(\mathbf{C})$ such that*

$$S_a^{-1} a = \infty. \tag{15}$$

*For every $b \in C_a(\Gamma')$, i.e., $b \in C(\Gamma')$ such that $b|a$, set*

$$S_b = \sigma_{ab} S_a. \tag{16}$$

Note the following features of the scaling matrix $S_a$. First, since $S_a$ is required to lie only in $\mathrm{SL}_2(\mathbf{C})$—not $\Gamma$—there is not only one—but a continuous family of—possible choices of $S_a$. Second, in our treatment there is no definite relationship between the group $\Gamma$ and the $S_a$ for $a \in C(\Gamma)$. *What relationship there is between the discrete groups and the choice of scaling matrices exists between different $b$ belonging to a fixed $C_a(\Gamma')$.* Namely, the somewhat arbitrary choice of the $S_a \in C(\Gamma)$ determines the $S_b$ for all $b \in C_a(\Gamma')$ via the formula (16). This comment extends to every object defined in terms of the $S_a$, such as the Eisenstein seres. The justification for our way of proceeding will become manifest in the simple form of the relationship we obtain between the Eisenstein series (Proposition (14)).

**Vector-valued Eisenstein Series.** Now we have assembled the various ingredients for the definitions of the $E_a$ and $E_b$. In the below, $z \in \mathbf{H}^3$ is of the form

$$z = x + y\mathbf{j}, \text{ with } x \in \mathbf{C},\ y > 0,$$

so that $y(z)$ denotes the 'y' coordinate of $z$, a positive real number.

**Definition 13** *Let* $a \in C(\Gamma)$, *and let the scaling matrix* $S_a$ *be defined as above. For* $\mathrm{Re} s > 2$, *the Eisenstein series is defined by*

$$E_a(z, s; \pi, \mathbf{w}) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} y^s(S_a^{-1}\gamma z)\pi(\gamma^{-1})\mathbf{w},\ a \in C(\Gamma),\ \mathbf{w} \in W_a. \tag{17}$$

*Similarly,*

$$E_b(z, s; \psi, \mathbf{v}) = \sum_{\gamma' \in \Gamma'_b \backslash \Gamma'} y^s(S_b^{-1}\gamma' z)\psi(\gamma'^{-1})\mathbf{v},\ b \in C(\Gamma'),\ \mathbf{v} \in V_b. \tag{18}$$

The Eisenstein series of (17) extend to elements of $\mathcal{A}(\Gamma, \pi, s)$, the space of $\pi$-automorphic forms of weight $s(2-s)$. Similarly, the Eisenstein series of (18) extend to Eisenstein elements of $\mathcal{A}(\Gamma, \pi, s)$. For the details of the theory of meromorphic continuation we refer to §6.1 of [6] for the $\mathbf{H}^3$, scalar case, and to §3.7 of [7] in the $\mathbf{H}^3$, vector case, to [5] for the $\mathbf{H}^2$, 'scalar' case. The methods of proof represented in these sources all have their origins in the work of Selberg [18].

Directly from Definition 13, we deduce the following two equivariance properties of the Eisenstein series.

**Equiv 1** $E_{\eta a}(z, s; \pi, \mathbf{w}) = E_a(z, s; \pi, \pi(\eta^{-1})\mathbf{w})$, for all $\eta \in \Gamma$.
**Equiv 2** $E_b(z, s; \pi, \mathbf{f}^a) = E_b(z, s; \pi, \mathbf{f}^b)$, for all $b|a$.

*Proof of equivariance properties.* For **Equiv 1**, one first notes that both sides do make sense because

$$W_{\eta a} = \pi(\eta^{-1})W_a.$$

We have from (17),

$$E_{\eta a}(z, s; \pi, \mathbf{w}) = \sum_{\gamma \in \Gamma_{\eta a} \backslash \Gamma} y^s(S_{\eta a}^{-1}\gamma z)\pi(\gamma^{-1})\mathbf{w} = \sum_{\gamma \in \Gamma_{\eta a} \backslash \Gamma} y^s(S_a^{-1}\eta^{-1}\gamma z)\pi(\gamma^{-1})\mathbf{w},$$

since $\sigma_{a,\eta a}$ is just $\eta$. Then we may change variables $\gamma \mapsto \eta\gamma$, to obtain

$$E_{\eta a}(z, s; \pi, \mathbf{w}) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} y^s(S_a^{-1}\gamma z)\pi(\gamma^{-1}\eta^{-1})\mathbf{w},$$

which one readily recognizes as

$$E_a(z, s; \pi, \pi(\eta^{-1})\mathbf{w}).$$

For **Equiv 2**, we have

$$\begin{aligned} E_a(a,s;\pi,\mathbf{f}^a) = & \sum_{\gamma\in\Gamma_a\backslash\Gamma} y^s(S_a^{-1}\gamma z)\pi(\gamma^{-1})\mathbf{f}^a \\ = & \sum_{\gamma\in\Gamma_a\backslash\Gamma} y^s(S_b^{-1}\sigma_{ab}\gamma z)\pi(\gamma^{-1})\mathbf{f}^a \\ = & \sum_{\gamma\in\Gamma_b\backslash\Gamma} y^s(S_b^{-1}\gamma z)\pi(\gamma^{-1})\pi(\sigma_{ab})\mathbf{f}^a \\ = & \sum_{\gamma\in\Gamma_b\backslash\Gamma} y^s(S_b^{-1}\gamma z)\pi(\gamma^{-1})\mathbf{f}^b \end{aligned}$$

where in the second-to-last line we have done a change of variables, setting the 'new' $\gamma$ equal to $\sigma_{ab}\gamma$, and in the last line we have used (14). □

**Proposition 14** *For $b|a$ and $\mathbf{f}\in V_b$, we have*

$$T: E_b(z,s;\psi,\mathbf{f}) = n_b^{1/2}E_a(z,s;\pi,\mathbf{f}^a). \tag{19}$$

*Proof.* Fix a coset representative

$$\alpha_i = b_{cj}\sigma_{ac}.$$

of $\Gamma'$ in $\Gamma$, in the above notation. Then we first claim that

$$TE_b(z,s;\psi,\mathbf{f})(\alpha) = E_b(\alpha z,s;\psi,\mathbf{f}). \tag{20}$$

The reason for (20) is that, by the definition of $T_a$, and $L_a$,

$$TE_b(z,s;\psi,\mathbf{f})(\alpha_i) = L_a\circ\oplus_k E_b(\alpha_k z,s;\psi,\mathbf{f})(\alpha_i) = E_b(\alpha z,s;\psi,\mathbf{f}).$$

Applying (18) to the right-hand side of (20), we obtain

$$TE_b(z,s;\psi,f) = \sum_{\eta\in\Gamma_b'\backslash\Gamma'} y^s(S_b^{-1}\eta\alpha_i z). \tag{21}$$

For readability of the formulas, from this point, we drop the subscript $i$ from $\alpha_i$, writing simply $\alpha$ instead. Note that, by **Equiv 2**, the right-hand side of (19) is

$$\begin{aligned} n_b^{1/2}E_b(z,s;\pi,\mathbf{f}^b)(\alpha) = & \; n_b^{1/2}\sum_{\gamma\in\Gamma_b\backslash\Gamma} y^s(S_b^{-1}\gamma z)\pi(\gamma^{-1})\mathbf{f}^b(\alpha) \\ = & \; n_b^{1/2}\sum_{\Gamma_b\backslash\Gamma} y^s(S_b^{-1}\gamma z)\mathbf{f}^b(\alpha\gamma^{-1}). \end{aligned}$$

By definition of $\mathbf{f}^b$,

$$\mathbf{f}^b(\alpha\gamma^{-1}) = n^{-1/2}(L_a(0,\ldots,0,\mathbf{f},\ldots,\mathbf{f},0,\ldots,0))(\alpha\gamma^{-1}).$$

If $\alpha\gamma^{-1} \in \Gamma'\beta_{bj}$ for some $j$, $1 \leq j \leq n_b$. Thus we have

$$\mathbf{f}^b(\alpha\gamma^{-1}) = \begin{cases} \psi(\alpha\gamma^{-1}\beta_{bj}^{-1})\mathbf{f} & \text{if } \alpha\gamma^{-1} \in \Gamma'\beta_{bj}^{-1},\ 1 \leq j \leq n_b \\ 0 & \text{otherwise.} \end{cases}$$

We therefore may write

$$\mathbf{f}^b(\alpha\gamma^{-1}) = \sum_{j=1}^{n_b} \tilde{\psi}(\alpha\gamma^{-1}\beta_{bj}^{-1})\mathbf{f},$$

where by definition $\tilde{\psi}(\cdot)$ is $\psi$ on $\Gamma'$ and 0 on the complement $\Gamma - \Gamma'$ of $\Gamma'$ in $\Gamma$. Since the $n_b^{-1/2}$ coming from the normalization of $\mathbf{f}^b$ cancels the $n_b^{1/2}$ we started with, up to this point we have shown that

$$n_b^{1/2} E_b(z, s; \pi, \mathbf{f}^b)(\alpha) = \sum_{\gamma \in \Gamma_b \backslash \Gamma} \sum_{j=1}^{n_b} y^s(S_b^{-1}\gamma z)\tilde{\psi}(\alpha\gamma^{-1}\beta_{bj}^{-1})\mathbf{f}.$$

By Proposition 5, the two summations on the right-hand side actually parametrize a single summation of a 'new' $\gamma$ over $\Gamma_b'\backslash\Gamma$. So up to this point, we have shown that

$$\begin{aligned} n_b^{1/2} E_b(z, s; \pi, \mathbf{f}^b)(\alpha) &= \sum_{\gamma \in \Gamma_b'\backslash\Gamma} y^s(S_b^{-1}\gamma z)\tilde{\psi}(\alpha\gamma^{-1})\mathbf{f} \\ &= \sum_{\gamma \in \Gamma_b'\backslash\Gamma'} \sum_{h \in \Gamma'\backslash\Gamma} y^s(S_b^{-1}\gamma z)\tilde{\psi}(\alpha h^{-1})\mathbf{f} \end{aligned}$$

But $\alpha h^{-1} \in \Gamma'$ if and only $\alpha = h$, since $h$ is ranging over coset representatives, and $\alpha$ is a fixed coset representative. In the case that $\alpha = h$, $\tilde{\psi}(\alpha h^{-1}) = \tilde{\psi}(1) = 1$. Therefore, we see that

$$n_b^{1/2} E_b(z, s; \pi, \mathbf{f}^b)(\alpha) = \sum_{\eta \in \Gamma_b'\backslash\Gamma'} y^s(S_b^{-1}\eta\alpha_i z),$$

which matches the right-hand side of (21). This completes the proof of Proposition 14. $\square$

As we will now explain, Proposition 14 implies equalities between various other spectral 'objects' associated to the Kleinian groups $\Gamma$ and $\Gamma'$.

**Determinant of Scattering matrix.** As noted in Theorem 3.5 of [12], the inductivity of the trace formula also implies that a number of other spectral objects closely related to the Selberg Zeta function satisfy the Artin formalism. Friedman, in [9], Theorem 5.1 proves that the determinant $\phi$ of the scattering matrix $\mathfrak{S}$, in the setting of this paper, satisfies the Artin formalism, but under the assumption of normality of $\Gamma_1$ in $\Gamma$. Since the scattering matrix is entirely an expression of the Fourier expansion of the Eisenstein series at the singular cusps, the Artin Formalism for the scattering matrix now follows immediately and in full generality (*i.e.*, without the normality assumption) from our Proposition 14.

To save space, we refer the reader to [4] for further details concerning the determinant of the scattering matrix. Our purpose in bringing this example up here is to point out that the Artin formalism for any spectral object which is expressible as "images" of the trace formula or parts of the trace formula is an immediate consequence of the termwise inductivity of the trace formula, whose proof we are engaged in and which will be completed in Theorem 20. Thus, in keeping with the vision of [12], the inductivity of the trace formula, or more precisely the 'pre'-trace formula, is the perspective unifying all aspects of the Artin formalism.

**Eisenstein Kernels.** In order to define these kernels, we have to explain some data associated to the cuspidal subgroups $\Gamma_a$. For each $a \in C(\Gamma)$, there is a lattice $\Lambda_a$ in $\mathbf{C}$ such that

$$\mathbf{c}(S_a)\Gamma_a^U = \begin{pmatrix} 1 & \Lambda_a \\ 0 & 1 \end{pmatrix} := \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \;\middle|\; \lambda \in \Lambda_\alpha \right\}$$

We let $|\Lambda_a|$ be the volume of the quotient $\Lambda_a \backslash \mathbf{C}$, or equivalently, the Euclidean measure of a fundamental parallelogram for $\Lambda_a$.

It is a new feature of the case of Kleinian groups (subgroups of $\mathrm{SL}(2, \mathbf{C})$) as opposed to Fuchsian groups (subgroups of $\mathrm{SL}(2, \mathbf{R})$) that the cuspidal subgroup can properly contain the *unipotent* cuspidal subgroup. Therefore, we have another piece of data, namely the index

$$[\Gamma_a : \Gamma_a^U] = 1, 2, 3, 4, \text{ or } 6.$$

The five possibilities for the index are the only ones allowed by the so-called 'crystallographic restriction'. The fundamental relation among these data and the width of the cusp $b \in C_a(\Gamma')$ is that

$$n_b = \frac{[\Gamma_a : \Gamma_a^U]}{|\Lambda_a|} \frac{|\Lambda_b|}{[\Gamma_b : \Gamma_b^U]}. \tag{22}$$

Now, we define the **Eisenstein kernel**

$$\mathcal{E}_\pi : \mathbf{H} \times \mathbf{H} \to \mathbf{C},$$

by

$$\mathcal{E}_\pi(z,z') = \sum_{a\in C(\Gamma)} \sum_{j=1}^{k_a(\pi)} \frac{[\Gamma_a : \Gamma_a^U]}{|\Lambda_a|} \frac{1}{4\pi} \int_{\mathbf{R}} \tilde{h}(t) E_a(z, 1+\mathbf{i}t; \pi, \mathbf{f}_j^a) \otimes \overline{E_a(z', 1+\mathbf{i}t; \pi, \mathbf{f}_j^a)}\, \mathrm{d}t, \tag{23}$$

where $\{\mathbf{f}_j^a\}$ is an orthonormal basis of $W_a$. Here, $\tilde{h}(t)$ is a function satisfying certain conditions to be specified precisely in §5, below. The exact nature of $\tilde{h}(t)$ does not matter at the moment. What matters is that the integral over $\mathbf{R}$ in the definition of $\mathcal{E}_\pi(z,z')$ converges, which we simply assume for the moment.

We have a definition of $\mathcal{E}_\psi(z,z')$ associated to $\Gamma'$ and the representation $(\psi, V)$, which is entirely analogous.

We study the Eisenstein kernel restricted to the diagonal, *i.e.*, the locus of $z = z'$. It is not difficult to verify that, if we set

$$f_\pi(z) := \mathcal{E}_\pi(z,z), \tag{24}$$

then

$$\mathrm{tr}_W f_\pi(z) = \sum_{a\in C(\Gamma)} \sum_{j=1}^{k_a(\pi)} \frac{[\Gamma_a : \Gamma_a^U]}{|\Lambda_a|} \frac{1}{4\pi} \int_{\mathbf{R}} \tilde{h}(t) |E_a(z, 1+\mathbf{i}t; \pi, \mathbf{f}_j^a)|_W^2\, \mathrm{d}t.$$

The automorphicity of the Eisenstein series allow us to deduce that,

$$f_\pi \in C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma}.$$

Similarly, if we set

$$f_\psi(z) := \mathcal{E}_\psi(z,z), \tag{25}$$

then we have

$$f_\psi \in C^\infty(\mathcal{E}_V)^{\mathrm{tr},\Gamma'}.$$

In fact, Proposition 14 implies the following.

**Corollary 15** *Wtih $f_\psi$, $f_\pi$ as given by* (24), (25) *above* ($f_\psi$, $f_\pi$) *forms an inductive pair in the sense of §2. As a consequence, we have*

$$\mathrm{tr}_V \mathrm{tr}_{\Gamma'}^{\Gamma} \mathcal{E}_\psi(z,z) = \mathrm{tr}_W \mathcal{E}_\pi(z,z), \text{ for all } z \in \Gamma\backslash G.$$

*Proof.* We are to show that $(f_\psi, f_\pi) = (\mathcal{E}_\psi, \mathcal{E}_\pi)$ satisfy (11), *i.e.*, we are supposed to show that

$$\mathrm{tr}_{\Gamma'}^{\Gamma} \mathcal{E}_\psi = \mathrm{tr}_V^W \mathcal{E}_\pi$$

We calculate that

$$\begin{aligned}
&\operatorname{tr}_{\Gamma'}^{\Gamma}\mathcal{E}_\psi(z,z) = \\
&\sum_{i=1}^{n}\sum_{b\in C(\Gamma')}\sum_{j=1}^{k_b(\psi)}\frac{[\Gamma_b:\Gamma_b^U]}{|\Lambda_b|}\frac{1}{4\pi}\int_{\mathbf{R}}\tilde{h}(t)E_b(\alpha_i z,1+\mathbf{i}t;\psi,\mathbf{f}_j^b)\otimes\overline{E_b(\alpha_i z,1+\mathbf{i}t;\psi,\mathbf{f}_j^b)}\,\mathrm{d}t, \\
&=\operatorname{tr}_V^W\sum_{b\in C(\Gamma')}\sum_{j=1}^{k_b(\psi)}\frac{[\Gamma_b:\Gamma_b^U]}{|\Lambda_b|}\frac{1}{4\pi}\int_{\mathbf{R}}\tilde{h}(t)TE_b(z,1+\mathbf{i}t;\psi,\mathbf{f}_j^b)\otimes\overline{TE_b(z,1+\mathbf{i}t;\psi,\mathbf{f}_j^b)}\,\mathrm{d}t \\
&=\operatorname{tr}_V^W\sum_{b\in C(\Gamma')}\sum_{j=1}^{k_b(\pi)}n_b\frac{[\Gamma_b:\Gamma_b^U]}{|\Lambda_b|}\frac{1}{4\pi}\int_{\mathbf{R}}\tilde{h}(t)E_a(z,1+\mathbf{i}t;\psi,(\mathbf{f}_j^b)^a)\otimes\overline{E_a(z,1+\mathbf{i}t;\psi,(\mathbf{f}_j^b)^a)}\,\mathrm{d}t
\end{aligned}$$

Applying (22) and Proposition 11(c), we deduce that

$$\begin{aligned}
&\operatorname{tr}_{\Gamma'}^{\Gamma}\mathcal{E}_\psi(z,z) = \\
&=\operatorname{tr}_V^W\sum_{a\in C(\Gamma)}\sum_{j=1}^{k_a(\pi)}\frac{[\Gamma_a:\Gamma_a^U]}{|\Lambda_a|}\frac{1}{4\pi}\int_{\mathbf{R}}\tilde{h}(t)E_a(z,1+\mathbf{i}t;\psi,\mathbf{f}_j^a)\otimes\overline{E_a(z,1+\mathbf{i}t;\psi,\mathbf{f}_j^a)}\,\mathrm{d}t \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \operatorname{tr}_V^W\mathcal{E}_\pi(z,z).
\end{aligned}$$

This completes the proof of the corollary. □

We cannot immediately say anything concerning integrals over the respective locally symmetric spaces, because the integrals of the Eisenstein kernels themselves do not converge. What does converge is the integral of a certain *difference* of the Eisenstein kernels and part (the 'cuspidally periodized' kernel $K^\psi_{\text{cusp}}$—see below) of the point-pair invariant. We will formulate these this notions in the next section.

## 5 Pre-Trace Formula

We are now working towards the statement of a "schematic" formulation (31) of the Selberg trace formula that we call the *pre-trace formula*.

**Point-pair invariants and the Selberg-Harish-Chandra transform.**

**Definition 16** *The* ***basic point-pair invariant*** $\boldsymbol{\delta}:\mathbf{H}\times\mathbf{H}\to[\mathbf{1},\boldsymbol{\infty})$ *is defined by*

$$\delta(z,z') = \frac{|x-x'|^2+y^2+{y'}^2}{2yy'} = \cosh(d(z,z')). \tag{26}$$

*For any $k\in\mathcal{S}([1,\infty))$, the Schwartz space, the* ***point-pair invariant*** $\boldsymbol{K}$

***associated to k*** *is a function on* $\mathbf{H} \times \mathbf{H}$ *defined by*

$$K(z, z') = k(\delta(z, z')), \ \textit{for}\, z, z' \in \mathbf{H}.$$

By (26), $K(z, z')$ depends only on the distance $d(z, z')$ between the points of the pair $(z, z') \in \mathbf{H} \times \mathbf{H}$. Hence, the terminology "point-pair invariant".

**Definition 17** *For a measurable function $k$ on $[1, \infty)$, define the **Selberg-Harish-Chandra transform $h : C \to C$ of $k$** by the integral*

$$h(\lambda) = \frac{\pi}{s} \int_1^\infty k\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right)\left(t^s - t^{-s}\right)\left(t - \frac{1}{t}\right)\frac{\mathrm{d}t}{t} \ \textit{for}\ s \neq 0, \qquad (27)$$

*when the integral exists, where $\lambda$ and $s$ are related by*

$$\lambda = 1 - s^2.$$

*In (27), for $s = 0$, replace $\frac{1}{s}\left(t^s - t^{-s}\right)$ by is limit $2\log t$.*

For the general theory of the Selberg-Harish-Chandra transform, see §3.5 of [6]. The only parts of the general theory that we need to recall here is that

- For $k \in \mathcal{S}([1, \infty))$, $h$ indeed exists (the integral (27) converges).
- In the spectral expansion (30), below, of the periodized kernel $K_\pi^\Gamma$ associated to $k$, and the Kleinian group $\Gamma$, the eigenvalues in the discrete spectrum of the Laplacian appear *as the arguments of the function $h$.*
- The function $\tilde{h}$ appearing inside $\mathcal{E}_\pi$ above is related to $h$ by

$$\tilde{h}(r) = h(1 + r^2).$$

Next, we give the *Selberg conditions*, which are sufficient conditions given in terms of the Selberg-Harish-Chandra transform $h : \mathbf{C} \to \mathbf{C}$ of $k$ for the pair $(h, k)$ to satisfy the pre-trace formula 31 below.

**Sel 1** $h$ is holomorphic in the strip $\{s \in \mathbf{C} \mid |\mathrm{Im}(s)| < 2 + \delta\}$ for some $\delta > 0$.
**Sel 2** $h(r) = O((1 + |r|^2)^{-3/2-\epsilon})$ in this strip.

**Partially-periodized kernels.** We continue in the setup of §2. Let $\Omega$ be any subset of $\Gamma$ which is closed under conjugation by elements of $\Gamma$. Then define

$$K_\pi^\Omega : \mathbf{H} \times \mathbf{H} \to \mathbf{C},$$

by

$$K_\pi^\Omega(z, z') = \sum_{\gamma \in \Omega} K(\gamma z, z')\pi(\gamma^{-1}). \qquad (28)$$

For example, we have $K_\pi^{\mathrm{Id}}$ when $\Omega = \{I_2\}$ (equal to the original kernel itself), and $K_\pi^\Gamma$, the fully periodized kernel. We also have $K_\pi^{\mathrm{hyp}}$, $K_\pi^{\mathrm{nce}}$ and $K_\pi^{\mathrm{cusp}}$ when

$\Omega$ is the the set of hyperbolic elements, noncuspidal elliptic elements, and cuspidal elements, respectively. (By definition, $\gamma$ is said to be hyperbolic when the absolute value of its trace as a matrix $|\mathrm{tr}(\gamma)| > 2$).

In contrast to the scalar case, the whole partially periodized kernel $K_\pi^\Omega$ lacks the property of $\Gamma$-invariance. However, if we 'restrict to the diagonal' and set

$$f_\pi^\Omega(z) := K_\pi^\Omega(z, z), \tag{29}$$

then we do have

$$f_\pi^\Omega \in C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma}.$$

We therefore define

$$F_\pi^\Omega = \mathrm{tr}_W f_\pi^\Omega \in C^\infty(\Gamma\backslash G).$$

Finally, when $F^\Omega$ is integrable, this allows us to define

$$I_\pi^\Omega = \int\limits_{\Gamma\backslash G} F_\pi^\Omega(z)\,\mathrm{d}z.$$

It is not particularly difficult to show—and it is part of the standard theory of Selberg—that the 'orbital' integrals $I_\pi^{\mathrm{Id}}$, $I_\pi^{\mathrm{hyp}}$, $I_\pi^{\mathrm{nce}}$ converge. See, for example, §5.2 of [6]. Although the integral of $F_\pi^{\mathrm{cusp}}(z)$ over $\Gamma\backslash\mathbf{H}$ does not converge, we can define

$$\tilde{f}_\pi^{\mathrm{cusp}} = f_\pi^{\mathrm{cusp}} - \mathcal{E}_\pi \in C^\infty(\mathcal{E}_W)^{\mathrm{tr},\Gamma}$$

It turns out that the *difference*

$$\tilde{F}_\pi^{\mathrm{cusp}} := \mathrm{tr}_W(f_\pi^{\mathrm{cusp}} - \mathcal{E}_\pi) = \mathrm{tr}_w f_\pi^{\mathrm{cusp}} - \mathrm{tr}_W \mathcal{E}_\pi$$

is indeed integrable on $\Gamma\backslash\mathbf{H}$. Demonstrating the integrability of $\tilde{F}_\pi^{\mathrm{cusp}}$ involves a fairly deep calculation of the asymptotics over compact "approximating domains". This calculation is one of the main ingredients of the trace formula, and it was first carried out by Selberg in the $\mathbf{H}^2$ case. For the corresponding calculations in the $\mathbf{H}^3$, scalar case see Proposition 6.5.3 of [6]. See §4.3 of [7] for the $\mathbf{H}^3$, vector case. A corresponding, much more general statement, is now part of the standard theory of the Arthur-Selberg trace formula for general reductive groups. See, *e.g.*, [1], Lectures 6–8.

**Spectral Expansion and Pre-trace formula.** The other main ingredient in the pre-trace formula is the spectral expansion of the fully periodized kernel, which gives

$$K_\pi^\Gamma(z, z') = \sum_j h(\lambda_j(\Gamma, \pi))\phi_j(z) \otimes \overline{\phi_j(z')} + \mathcal{E}_\pi(z, z'), \tag{30}$$

for which see Chapter 4 of [7] in this particular case, and §6.3 of [6] and Chapter IV of [19] for similar cases.

Re-arranging the spectral expansion, writing $\Gamma$ as a union of conjugacy classes, and then restricting to the diagonal by setting $z = z'$, we have

$$\sum_j h(\lambda_j(\Gamma,\pi))\phi_j \otimes \overline{\phi_j} = f_\pi^{\mathrm{Id}} + f_\pi^{\mathrm{hyp}} + f_\pi^{\mathrm{nce}} + \tilde{f}_\pi^{\mathrm{cusp}}.$$

Applying $\mathrm{tr}_W$ of both sides results in the equality of functions,

$$\sum_j h(\lambda_j(\Gamma,\pi))|\phi_j|^2 = F_\pi^{\mathrm{Id}} + F_\pi^{\mathrm{hyp}} + F_\pi^{\mathrm{nce}} + \tilde{F}_\pi^{\mathrm{cusp}}.$$

Integrating (and, of course, using the integrability of each of the terms on the right side) yields the equality

$$\sum_j h(\lambda_j(\Gamma,\pi)) = I_\pi^{\mathrm{Id}} + I_\pi^{\mathrm{hyp}} + I_\pi^{\mathrm{nce}} + \tilde{I}_\pi^{\mathrm{cusp}}. \tag{31}$$

We refer to (31) as the **pre-trace formula associated to** $(\Gamma,\pi)$. Further, more explicit, evaluation of the integrands, as undertaken by Selberg, Venkov in [22], and §§5.2 and 6.5 of [6], results in what is properly referred to as *the trace formula associated to the pair* $(\Gamma,\pi)$. However, the 'pre-trace' formula, (31) will suffice for our purposes.

Naturally, we can formulate the an analogous pre-trace formula for $(\Gamma',\psi)$, as well as the corresponding kernels, and restrictions and their traces. We have already established an equality, in Proposition 10, between the 'spectral', *i.e.* left-hand, sides of two the pre-trace formulas for $(\Gamma,\pi)$ and $(\Gamma',\psi)$, and also an equality for part (namely the Eisenstein integral part) of the cuspidal integral on the 'geometric', *i.e.*, right-hand, side. The main result of the next section, for which we return to the notion of an inductive pair of sections, completes this development by establishing equalities between all the remaining pairs of 'corresponding' terms in the two trace formulas.

## 6 Inductivity and Applications

**Inductive pairs of subsets.** For this paragraph we maintain the notation established in the Sections 2–5.

**Definition 18** *A pair* $(\Omega',\Omega)$ *of conjugation-invariant subsets of* $(\Gamma',\Gamma)$ *is said to form an* ***inductive pair of subsets*** *if for every choice of* $k \in \mathcal{S}([1,\infty))$*, the pair of 'diagonal restrictions'*

$$(f_\psi^{\Omega'}, f_\pi^{\Omega}) \in C^\infty(\mathcal{E}_\psi)^{\mathrm{tr},\Gamma'} \times C^\infty(\mathcal{E}_\pi)^{\mathrm{tr},\Gamma},\ \textit{defined as in}\,(29),$$

*forms an inductive pair of sections in the sense of §2.*

**Proposition 19** *Assume that* $(\Omega, \Omega')$ *is a pair of conjugation invariant subsets of* $(\Gamma, \Gamma')$ *such that*

$$\Omega' = \Omega \cap \Gamma.$$

*Then* $(\Omega, \Omega')$ *form an inductive pair of subsets of* $(\Gamma, \Gamma')$.

*Proof.* We calculate that for $z \in \mathbf{H}$, and arbitrary $k \in \mathcal{S}([1, \infty))$,

$$\begin{aligned}
\operatorname{tr}_{\Gamma'}^{\Gamma} f_{\psi}^{\Omega'}(z) &= \sum_{i=1}^{n} \sum_{\gamma \in \Omega'} K(\gamma \alpha_i z, \alpha_i z) \psi(\gamma^{-1}) \\
&= \sum_{i=1}^{n} \sum_{\gamma \in \Omega} K(\alpha_i^{-1} \gamma \alpha_i z, z) \tilde{\psi}(\gamma^{-1}) \text{ (by (5))} \\
&= \sum_{i=1}^{n} \sum_{\gamma \in \alpha_i^{-1} \Omega \alpha_i} K(\gamma z, z) \tilde{\psi}(\alpha_i \gamma^{-1} \alpha_i^{-1}) \text{ (change of variable } \gamma \mapsto \alpha_i \gamma \alpha_i^{-1}) \\
&= \sum_{i=1}^{n} \sum_{\gamma \in \Omega} K(\gamma z, z) \tilde{\psi}(\alpha_i \gamma^{-1} \alpha_i^{-1}) \text{ (conjugation-invariance of } \Omega) \\
&= \operatorname{tr}_{V}^{W} \sum_{\gamma \in \Omega} K(\gamma z, z) \pi(\gamma^{-1}) \text{ (by (4))} \\
&= \operatorname{tr}_{V}^{W} f_{\pi}^{\Omega}(z).
\end{aligned}$$

Thus $(f_{\psi}^{\Omega'}, f_{\pi}^{\Omega}) \in C^{\infty}(\mathcal{E}_V)^{\mathrm{tr},\Gamma'} \times C^{\infty}(\mathcal{E}_W)^{\mathrm{tr},\Gamma}$ forms an inductive pair of sections. This demonstrates that $(\Omega, \Omega')$ is an inductive pair of subsets. □

**Remarks.**

- The nature of the function $k$ used to define $K$ plays no role in the proof of Proposition 19—for example $K$ can even be replaced by the "basic" point pair invariant $\delta$, without affecting the argument in any way.
- Note that there is in Proposition 19 no normality assumption on the subgroup $\Gamma'$. Only the *subsets* $\Omega'$ and $\Omega$ are assumed to be conjugation-invariant in the respective group $\Gamma$ and $\Gamma'$.

**Artin Formalism Method I: Green's Kernel.** Consider the following function, or more precisely, family of functions indexed by a complex parameter $s$, with $\mathrm{Re}(s) > 1$, and taking argument $\delta \in (1, \infty)$,

$$\Phi_s(\delta) = \frac{(\delta + \sqrt{\delta^2 - 1})^{-s}}{\sqrt{\delta^2 - 1}} = \frac{\exp(-s \cosh^{-1} \delta)}{\sqrt{\delta^2 - 1}}.$$

Because of the rapid decay of $\Phi_s$, we can apply Definitions 16 and 17 with $k = \Phi_s$ to obtain the associated kernel

$$K_{\Phi_s}(z, z') = \Phi_s(\delta(z, z')),$$

and Selberg-Harish Chandra transform $h_{\Phi_s}$. The following facts concerning $h_{\Phi_s}$ are well-known

**$\Phi_s$ 1** The transform $h_{\Phi_s}$ does *not* satisfy the Selberg conditions **Sel 1–2**, and therefore $K_{\Phi_s}$ *not* admissible as a kernel for the pre-trace formula.

**$\Phi_s$ 2** The Fourier transform (see (33), below) of $\tilde{h}_{\Phi_s}$ can be computed, and we have

$$g_s(x) = \frac{2\pi}{s} e^{-s|x|}.$$

**$\Phi_s$ 3** Let $I_\pi^{\text{hyp}}(s)$ be the orbital integral associated to the kernel $K_{\Phi_s}(z, z')$ and the hyperbolic elements of $\Gamma$, as in §5 above. Then **$\Phi_s$ 2** and Proposition 22, below, imply that for $\mathrm{Re}(s) > 1$,

$$I_\pi^{\text{hyp}}(s) = \frac{2\pi}{s} \mathrm{d} \log Z_{\Gamma,\pi}(s).$$

The shortest route to obtaining the Artin formalism of the Selberg zeta function for general commensurable pairs of Kleinian groups uses **$\Phi_s$ 3**. From the discussion of the last part of §2, Proposition 19 implies that, in the above notation, $I_\psi^{\text{hyp}}(s) = I_\pi^{\text{hyp}}(s)$ for $\mathrm{Re}(s) > 1$, and then **$\Phi_s$ 3** and meromorphic continuation complete the proof of the Artin formalism of the zeta function. However, because of **$\Phi_s$ 1**, this method does not fit into the present paper's over-arching theme of seeing all aspects of the Artin Formalism as 'images' of the inductivity of the terms of the trace formula. We therefore say no more about the Green's function method here, and refer the reader to our other paper [4] for the details.

**Equality of pairs of terms in the pre-trace formula.** Now we apply the result of this section to all of the terms appearing in §5.

**Theorem 20** *All of the following are inductive pairs of sections.*

*(a)* $f_\psi(z) = K_\psi^{\Omega'}(z, z)$, $f_\pi(z) = K_\pi^{\Omega}(z, z)$ *for* $(\Omega', \Omega)$ *the hyperbolic, or identity, or noncuspidal elliptic elements of* $(\Gamma', \Gamma)$.

*(b)* $\tilde{f}_\psi^{\text{cusp}}(z) = K_\psi^{\text{cusp}}(z, z) - \mathcal{E}_\psi(z, z)$, $\tilde{f}_\pi^{\text{cusp}}(z) = K_\pi^{\text{cusp}}(z, z) - \mathcal{E}_\pi(z, z)$.

*(c)* $f_\psi^{\text{disc}} = \sum_j h(\lambda_j) \phi_j \otimes \overline{\phi_j}$, $f_\pi^{\text{disc}} = \sum_j h(\lambda_j) \Phi_j \otimes \overline{\Phi_j}$.

*Proof.* Part (a) follows immediately from Proposition 19. One applies the proposition with $\Omega$ equal to the set hyperbolic, identity, or noncuspidal elliptic elements of $\Gamma$.

Part (b) follows from Proposition 19 applied with $\Omega$ equal to the set of cuspidal elliptic elements, from Corollary 15, and from the evident observation that the difference of two pairs of inductive pairs of sections is an inductive pairs of sections.

Part (c) simply rephrases Part (b) of Proposition 10. □

Using the formalism introduced at the end of Section 2, we derive the following more 'familiar' forms of the equalities in Theorem 20, essentially images of these equalities under certain 'trace' maps.

**Corollary 21** *We have the following equalities*

*(a) For* $\Omega \in \{\text{hyp, nce, Id}\}$*,*

$$\mathrm{tr}^{\Gamma}_{\Gamma'}|\phi_j| = |\Phi_j|,\ \mathrm{tr}^{\Gamma}_{\Gamma'}\tilde{F}^{\text{cusp}}_{\psi} = \tilde{F}^{\text{cusp}}_{\pi},\ \mathrm{tr}^{\Gamma}_{\Gamma'}F^{\Omega}_{\psi} = F^{\Omega}_{\pi}.$$

*(b) For* $\Omega \in \{\text{hyp, nce, Id}\}$*,*

$$\sum_j h(\lambda_j(\Gamma,\pi)) = \sum_j h(\lambda_j(\Gamma',\psi)),\ \tilde{I}^{\text{cusp}}_{\pi} = \tilde{I}^{\text{cusp}}_{\psi},\ I^{\Omega}_{\pi} = I^{\Omega}_{\psi}.$$

*In other words, in the pre-trace formulas for* $\Gamma$ *and* $\Gamma'$*, each of the corresponding pairs of terms are equal.*

Readers should compare part (a) of Corollary 21 to the relation **A4. Induction.** in §2 of [12], which is the corresponding inductivity statement in the context of quotients of compact manifolds discussed in that article.

**Artin Formalism Method II: Heat Kernel.** In contrast to Green's Kernel, the heat kernel is suitable for use as a test function in trace formula. However, the use of the heat kernel does not give the 'additive' Selberg zeta function directly. Rather, the heat kernel, through the hyperbolic contribution to the trace formula, gives a certain integral transform of this zeta function.

In this section *only* we use the notation $k_t$ to refer to a specific family of test functions

$$k_t \circ \cosh^{-1} = e^{-t\Delta_{G/K}},\ \text{the heat family of gaussians.} \tag{32}$$

The point-pair invariant $K_t$ associated to the heat family of Gaussians $k_t$ is called the **heat kernel**. For an explanation of this terminology and a general introduction to the heat kernel we refer to [2]. For explicit formulas (which we will not use) and other characterizations of the heat kernel in the specific context of hyperbolic space we refer to [13] [12], [14]. Among the analytic properties of the heat kernel, the one which interests us in the current context is *quadratic exponential decay*. In particular, the decay property of the $k_t$ is more than sufficient to imply that the associated $h_t$ satisfies Selberg's conditions. Thus, $K_t$ is a suitable kernel for the pre-trace formula.

In order to make the connection between the heat kernel in the trace formula and the Selberg zeta function, we will need the following standard identity from the Selberg theory, the explicit evaluation of the hyperbolic contribution to the trace formula of $\Gamma$. In order to state Proposition 22 we make use of both the notation for terms of the trace formula introduced in §5 and the notation used to define the Selberg zeta function and state its logarithmic derivative in §1

**Proposition 22** *Using the notation for objects arising in trace formula introduced in §5, we have*

$$I_\pi^{\mathrm{hyp}} = 2 \sum_{\{P_0\}_\Gamma} \sum_{n=1}^{\infty} \frac{\log N(P_0) \mathrm{tr}_W \pi(P_0^n)}{m(P_0)|N(P_0)^{n/2} - N(P_0)^{-n/2}|^2} g(n \log N(P_0)),$$

*where*

- $\{P_0\}_\Gamma$ *is the set of primitive hyperbolic conjugacy classes in* $\Gamma$.
- $g$ *is the Fourier transform of* $\tilde{h}$, *i.e.*,

$$g(x) = \frac{1}{2\pi} \int_{\mathbf{R}} \tilde{h}(t) e^{-\mathbf{i}tx} \mathrm{d}t. \tag{33}$$

*and the remainder of the notation has been explained above.*

For the computation of $I_\pi^{\mathrm{hyp}}$, see, *e.g.*, Theorem 5.2.2 of [6]. There, the authors consider exclusively the case of $\pi$ the trivial one-dimensional representation, in which case the factor $\mathrm{tr}_W \pi(P^n)$ reduces to 1. The modifications needed to consider the general case are found in §4.3 of [7].

**Definition 23** *With the heat kernel $K_t$ as the test kernel, the hyperbolic contribution $I_\pi^{\mathrm{hyp}}$ to the trace formula is called the* ***theta function $\vartheta_\pi$ associated to $(\Gamma, \pi)$.***

We now combine Proposition 22 with the explicit formula

$$g_t(x) = \left(\frac{1}{4\pi t}\right)^{1/2} e^{-x^2/4t},$$

associated, via (27) and (33) to the family of heat Gaussians $k_t$ on the manifold $G/K$. We obtain the evaluated form of $\vartheta_\pi$,

$$\vartheta_\pi(t) = \frac{1}{(\pi t)^{1/2}} \sum_{\{P_0\}_\Gamma} \sum_{n=1}^{\infty} \frac{\log N(P_0) \mathrm{tr}_W \pi(P_0^n)}{m(P_0)|N(P_0)^{n/2} - N(P_0)^{-n/2}|^2} \exp(-(n \log N(P_0))^2/4t),$$

with all notation as in Proposition 22. Compare the formulas on the bottom of p. 233 of [15], as well in §VIII.5 of [14].

**Proposition 24** *The logarithmic derivative of the Selberg zeta function is related to the Gauss transform of $\vartheta_{\Gamma,\pi}(t)$ by*

$$\mathrm{d}\log Z_{\Gamma,\pi}(s)/\mathrm{d}s = \frac{s}{2}(\mathrm{Gauss}(\vartheta_{\Gamma,\pi}))(s) := \frac{s}{2}\int_0^\infty e^{-s^2 t}\vartheta_{\Gamma,\pi}(t)\,\mathrm{d}t. \tag{34}$$

*Proof.* Point by point, our calculations are formally analogous to [15], p. 239–40, but with the sums changed in order to be appropriate to the Kleinian, rather than Fuchsian case. Note that [15] uses $\chi_\pi$ to denote that trace that we write more explicitly as $\mathrm{tr}_W\pi$. Also, throughout our calculations, $\{P\}_\Gamma$ denotes a complete system of non-conjugate elements of $\Gamma$, $\{P_0\}$ denotes an associated complete system of non-conugate *primitive elements*, and for for $P$ in the former, $P_0$ denotes the associated element of the latter, *i.e.*, the unique primitive element $P_0$ such that $P = P_0^n$ for some integer $n$. In the range of convergence $\mathrm{Re}(s) > 1$, we calculate that

$$\begin{aligned}
\mathrm{d}\log Z_{\Gamma,\pi}(s)/\mathrm{d}s &= \sum_P \frac{\mathrm{tr}_W\pi(P)\log NP_0}{m(P)|\lambda(P)-\lambda(P)^{-1}|^2}N(P)^{-s} \\
&= \sum_{P_0 \text{ primitive}}\sum_{n=1}^\infty \frac{\mathrm{tr}_W\pi(P_0^n)\log NP_0}{m(P_0)|\lambda(P_0)-\lambda(P_0)^{-1}|^2}(|\lambda(P_0)|^2)^{-ns} \\
&= \sum_{P_0 \text{ primitive}}\sum_{n=1}^\infty \frac{\mathrm{tr}_W\pi(P_0^n)\log(|\lambda(P_0)|^2)}{m(P_0)|\lambda(P_0^n)-\lambda(P_0^n)^{-1}|^2}\left[(|\lambda(P_0)|^2)^{-s}\right]^n \\
&= \sum_{P_0 \text{ prim}}\sum_{n=1}^\infty \frac{\mathrm{tr}_W\pi(P_0^n)\log N(P_0)}{m(P_0)|\lambda(P_0^n)-\lambda(P_0^n)|^{-1}|^2}\times \\
&\qquad \times s\int_0^\infty e^{-s^2 t}(4\pi t)^{-1/2}e^{-\frac{(n\log N(P_0))^2}{4t}}\,\mathrm{d}t \\
&= \frac{s}{2}\int_0^\infty e^{-s^2 t}\vartheta_{\Gamma,\pi}(t)\,\mathrm{d}t.
\end{aligned}$$

Since $\mathrm{d}\log Z(s)$ extends to a meromorphic function on $\mathbf{C}$, the equality extends from the half-plane $\mathrm{Re}s > 1$ to give an equality of meromorphic functions. □

The problem of meromorphic continuation of $Z_{\Gamma,\pi}$, as opposed to $\mathrm{d}\log Z_{\Gamma,\pi}$ to the entire complex plane is more delicate than in the Fuchsian case. Friedman has shown in [8] that the residues of $\mathrm{d}\log Z_{\Gamma,\pi}$ can fail to be integers in certain circumstances (for certain choices of $\Gamma, \pi$). More precisely, he has shown in the case when $\Gamma$ has a single cusp at $\infty$ (combining material from Lemmas 6.7, 6.8, 6.10 of [8]),

- Suppose that $[\Gamma_\infty : \Gamma_\infty^U] = 1$ or 2. Then the poles of $\mathrm{d}\log Z_{\Gamma,\pi}$ are simple with integral residues, and consequently $Z_{\Gamma,\pi}$ has meromorphic continuation to $\mathbf{C}$.
- Suppose that $[\Gamma_\infty : \Gamma_\infty^U] = 3$. Then the poles of $\mathrm{d}\log Z_{\Gamma,\pi}$ are rational with denominators occurring up to 6, and consequently $Z_{\Gamma,\pi}^6$ (and in certain cases, no smaller power of $Z_{\Gamma,\pi}$) has meromorphic continuation to $\mathbf{C}$.

- Suppose that $[\Gamma_{,\infty} \colon \Gamma_\infty^U] = 4$. Then for some integer $N$, $Z_{\Gamma,\pi}^N$ is a meromorphic function, but there is no known universal bound on $N$ (power $N$ sufficing for all choices of $\Gamma, \pi$).
- Suppose that $[\Gamma_\infty : \Gamma_\infty^U] = 6$ (the last possibility allowed by the crystallographic restriction). Then it is *conjectured* that for some $N$, depending on $\Gamma, \pi$, $Z_{\Gamma,\pi}^N$ is a meromorphic function on $\mathbf{C}$.

Incorporating these caveats about the meromorphicity of $Z_{\Gamma,\pi}$, we can deduce from our main results the following generalization of the Artin formalism of $Z_{\Gamma,\pi}$.

**Corollary 25** *For $\Gamma, \Gamma'$ Kleinian groups, i.e., discrete cofinite subgroups of $\mathrm{SL}_2(\mathbf{C})$ containing $\{\pm 1\}$ such that*

$$\Gamma' \subseteq \Gamma \textit{ with } [\Gamma : \Gamma'] = n < \infty,$$

*we have*

$$\mathrm{d}\log_{\Gamma,\pi}(s) = \mathrm{d}\log_{\Gamma',\psi}(s),$$

*understood as an equality of meromorphic functions. Consequently, when for some $N \geq 1$, $Z_{\Gamma,\pi}^N$ (equivalently $Z_{\Gamma',\psi}^N$) has a meromorphic continuation to all $s \in \mathbf{C}$, we deduce the equality of meromorphic functions*

$$Z_{\Gamma,\pi}^N = Z_{\Gamma',\psi}^N.$$

*Proof.* By specializing Corollary 21(b) to the case $\Omega = \mathrm{hyp}$, we obtain the equality $\vartheta_{\Gamma,\pi} = \vartheta_{\Gamma',\psi}$. Under the Gauss transform of Proposition 24, the equality of theta functions transforms to the equality of logarithmic derivatives of Selberg zeta functions. □

## 7 Application: $Z_\Gamma(s, \chi)$ for $\Gamma = \mathrm{SO}(3, \mathbf{Z}[\mathbf{i}])$ and $\mathrm{SO}(2,1)_{\mathbf{Z}}$

In light of some results in [3], we can use Corollary 25 to relate the Selberg zeta function of the quotient of $\mathrm{SO}(3, \mathbf{C})$ by its full ring of integer points, for certain representations, to the 'standard' Selberg zeta function of $\mathrm{SL}(2, \mathbf{C})$ modulo its integer points, for certain related representations. In order to set this up we recall the results announced in §2 of [3].

As explained in §2 of [3], $\mathrm{SO}(3, \mathbf{Z}[\mathbf{i}])$ has a model as the elements of $\mathrm{Aut}(\mathfrak{sl}_2(\mathbf{C}))$ preserving the $\mathbf{Z}$-module on a basis of $\mathfrak{sl}_2(\mathbf{C})$ which is orthonormal for the

Killing form $B$. For definiteness, one fixes the $B$-orthonormal basis of $\mathfrak{sl}_2(\mathbf{C})$

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then setting $\beta = \mathbf{Z}\text{-span}\{X_1, X_2, X_3\}$, with the subgroup of $G = \mathrm{SO}(B)$, we identify the subgroup preserving the lattice $\beta$ with $\Gamma = \mathrm{SO}_3(\mathbf{Z}[\mathbf{i}])$. Taking for granted that this model is intended, we will henceforth write $G = \mathrm{SO}_3(\mathbf{C})$ for $\mathrm{SO}(B)$ and $\Gamma = \mathrm{SO}_3(\mathbf{Z}[\mathbf{i}])$ for the discrete subgroup preserving $\beta$.

We use $\mathbf{c}(\cdot)$ to denote that *conjugation map* for matrices.

**Proposition 26** *The map* $\mathbf{c}(\cdot)$ *induces an isomorphism of Lie groups,*

$$\mathrm{SL}_2(\mathbf{C})/\{\pm I\} \xrightarrow{\cong} G.$$

*Proof (sketch).* The map $\mathbf{c}(\cdot)$ is a concrete realization of the adjoint map $\mathrm{Ad} : \mathrm{SL}_2(\mathbf{C}) \to \mathrm{Aut}\, \mathfrak{sl}_2(\mathbf{C})$. It is well-known that for any semisimple Lie group $G$, Ad provides a embedding of $G/Z(G)$ into $\mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$ and $Z(G)$ is the center of $G$. Further, as is well known, $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{O}(B)$, the group of transformations of $\mathfrak{g}$ preserving the Killing form. Thus, the abstract argument says that $\mathbf{c}$ induces a morphism into $\mathrm{O}(B)$. Comparison of dimensions shows that the morphism is an isomorphism onto $\mathrm{SO}(B) = G$. □

As a result of Proposition 26, we see that the inverse image $\mathbf{c}^{-1}(\Gamma)$ is a Kleinian group containing $\{\pm 1\}$.

In [3], Proposition 2.1, the isomorphism $\mathbf{c}(\cdot)$ is calculated in terms of coordinates, namely the standard coordinates on $\mathrm{SL}_2(\mathbf{C})$, and the coordinates on $G$ induced by the basis $\beta$. Via some lengthy but not difficult calculations, the coordinatized version of $\mathbf{c}(\cdot)$ can be used to obtain a completely explicit description of the matrices in $\mathbf{c}^{-1}(\Gamma)$. In order to give this description, we introduce the following notation. Set

- $\omega_8 = \frac{\sqrt{2}}{2}(1+\mathbf{i})$, a primitive eighth root of unity that we can fix for definiteness.
- $\Xi = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}[\mathbf{i}]) \mid \gamma^2 = I_2 \mod (1+\mathbf{i})\}$.
- $\eta = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.
- $\alpha^N(m,x) = \begin{pmatrix} m & x \\ 0 & \frac{N}{m} \end{pmatrix}$, for $m, x, N \in \mathbf{Z}[\mathbf{i}]$ such that $m|N$.

Note that the element $\eta$ lies in the complement of $\Xi$ in $\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$, so that, in particular, $\Xi\eta$ is a nonidentity $\Xi$-coset in $\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$. Note also that while $\alpha$ is a matrix with integral entries (and even an upper-triangular one), $\alpha$ has determinant $N$. Unless $N$ is a unit, there is no natural way to view $\alpha$ itself as an element of a group in our setup. In practice, $\alpha$ will appear in our results multiplied by a factor, such that the determinant of the product, as a whole, *is* a unit.

**Proposition 27** *With $\Gamma = \mathrm{SO}_3(\mathbf{Z}[\mathbf{i}])$ as above, $\mathbf{c}$ the conjugation map as in Proposition 26, we have the following explicit description of the inverse image $\mathbf{c}^{-1}(\Gamma)$ as a Kleinian group containing $\{\pm 1\}$,*

$$\mathbf{c}^{-1}(\Gamma) = \bigcup_{\delta=0,1} \left( \Xi \left( \frac{1}{\omega_8^\delta} \alpha^{\mathbf{i}^\delta}(\mathbf{i}^\delta, 0) \right) \bigcup \left( \bigcup_{\epsilon=0,1} \Xi\eta \left( \frac{1}{\omega_8^\delta(1+\mathbf{i})} \alpha^{2\mathbf{i}^{1+\delta}}(\mathbf{i}^{1+\delta}, \mathbf{i}^\epsilon) \right) \right) \right),$$

*with all sets in the above union disjoint.*

We restate Proposition 27 in a form better adapted for the application of Corollary 25.

- The intersection $\mathbf{c}^{-1}(\Gamma) \cap \mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$ is equal to $\Xi$.
- The group $\Xi$ has index 6 in the group $\mathbf{c}^{-1}(\Gamma)$ and has index 3 in the group $\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$.
- For a complete set of (right) coset representatives for $\Xi$ in $\mathbf{c}^{-1}(\Gamma)$, we may take
$$\left\{ \begin{matrix} I_2, & \frac{1}{\omega_8}\alpha^{\mathbf{i}}(\mathbf{i}, 0), & \frac{1}{1+\mathbf{i}}\eta\alpha^{2\mathbf{i}}(\mathbf{i}, 1), \\ \frac{1}{1+\mathbf{i}}\eta\alpha^{2\mathbf{i}}(\mathbf{i}, \mathbf{i}), & \frac{1}{\omega_8(1+\mathbf{i})}\eta\alpha^{-2}(-1, 1), & \frac{1}{\omega_8(1+\mathbf{i})}\eta\alpha^{-2}(-1, \mathbf{i}) \end{matrix} \right\},$$
while a complete set of coset representatives for $\Xi$ in $\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$ is
$$\left\{ I_2, \eta, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \tag{35}$$

Given any finite dimensional unitary representation $(\psi, V)$ of $\Xi$, we may use the explicit list of coset representatives in concert with (4)–(5) to compute, explicity, models for the induced representations $(\pi, V^3)$ on $\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$ and $(\rho, V^6)$ on $\mathbf{c}^{-1}(\Gamma)$. Then, by Corollary 25 we have

$$\mathrm{d}\log Z_{\mathbf{c}^{-1}(\Gamma),(\rho,V^6)} = \mathrm{d}\log Z_{\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}]),(\pi,V^3)}.$$

By Proposition 27, we may take the preceding relation as a definition of the left-hand side (*i.e.*, $\mathrm{d}\log Z_{\mathbf{c}^{-1}(\Gamma),(\rho,V^6)}$). Further, since $\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])$ has precisely one cusp, at $\infty$, and the index of the unipotent stabilizer of $\infty$ in the stabilizer is 2, the results of Friedman surveyed just before Corollary 25 imply that the residues of the right-hand side (outside the range of convergence) are integral, so that the equality of logarithmic derivatives leads to an equality of meromorphic functions, which we codify as follows.

**Definition 28** *Let* $\Gamma = \mathrm{SO}_3(\mathbf{Z}[\mathbf{i}])$*, and let the rest of the notation in this section be as above. Let* $\pi = \mathrm{Ind}_{\Xi}^{\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}])}\psi$*, and* $\rho = \mathrm{Ind}_{\Xi}^{\mathbf{c}^{-1}(\Gamma)}\psi$*, with* $\rho$ *considered as a representation of* $\Gamma$ *via the isomorphism* $\mathbf{c}(\cdot)$*. Then, in accordance with the Artin formalism as described in this section, we define the meromorphic function* $Z_{\Gamma,\rho}$ *by*

$$Z_{\Gamma,\rho} := Z_{\mathrm{SL}_2(\mathbf{Z}[\mathbf{i}]),\pi}.$$

Further, defining the Fuchsian groups

$$\Gamma_{\mathbf{Z}} := \mathrm{SO}(2,1)_{\mathbf{Z}} = \Gamma \cap \mathrm{GL}_3(\mathbf{R}),$$

and

$$\Xi_{\mathbf{Z}} = \Xi \cap \mathrm{SL}_2(\mathbf{R}),$$

as the corresponding discrete subgroups of the 'split' real forms, we can carry out a similar, but somewhat simpler, sort of analysis to that concerning the pair $(\Xi, \Gamma)$. Then it turns out that (summarizing Propositions 5.5 and Lemma 5.6 of [3]),

- The intersection $\mathbf{c}^{-1}(\Gamma_{\mathbf{Z}}) \cap \mathrm{SL}_2(\mathbf{Z}) = \Xi_{\mathbf{Z}}$.
- The indices of $\Xi_{\mathbf{Z}}$ in $\mathbf{c}^{-1}(\Gamma_{\mathbf{Z}})$ and $\mathrm{SL}_2(\mathbf{Z})$ are 2 and 3, respectively.
- A representative of the unique non-identity right coset of $\Xi_{\mathbf{Z}}$ in $\mathbf{c}^{-1}(\Gamma)$ is given by $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$.

In parallel with Definition 28, we may define, in accordance with the Artin formalism for Fuchsian groups,

$$Z_{\Gamma_{\mathbf{Z}},\rho} := Z_{\mathrm{SL}_2(\mathbf{Z}),\pi}, \tag{36}$$

where $\rho$ and $\pi$ are the representations induced from a finite-dimensional unitary representation $\psi$ on $\Xi_{\mathbf{Z}}$.

Owing to the normality of $\Xi_{\mathbf{Z}}$ in $\Gamma_{\mathbf{Z}}$ and the particularly simple description of the representations of the finite abelian quotient groups we obtain

$$\mathbf{Z}_2 \cong (\Xi_{\mathbf{Z}} \backslash \Gamma_{\mathbf{Z}})^* = \{\mathbf{1}, \mathrm{sgn}\},$$

where sgn denotes the sign representation of $\mathbf{Z}_2$ considered as a one-dimensional representation of $\Gamma_{\mathbf{Z}}$ trivial on $\Xi_{\mathbf{Z}}$. Thus, taking the special case of (36) when $\psi = \mathbf{1}$, the trivial one-dimensional representation of $\Xi_{\mathbf{Z}}$, we obtain the cute formula

$$Z_{\Gamma} \cdot Z_{\Gamma,\mathrm{sgn}} = Z_{\mathrm{SL}_2,\mathrm{Ind}_{\Xi}\mathbf{1}}. \tag{37}$$

In (37), in order to improve readability, we have omitted the subscript $\mathbf{Z}$, throughout, and used the obvious identification of $Z_{\Gamma,1}$ with Selberg's original 'scalar' zeta function.

## 8 Comments on higher dimensions

Since the group $\mathrm{SO}(2,1)$ is the smallest in an infinite sequence of real Lie groups of rank one, but increasing dimension, namely the hyperbolic isometry groups $\mathrm{SO}(n,1)$, consideration of the example of §7 naturally leads to the question of whether an analogue of the Artin formalism for Selberg zeta functions for arbitrary lattices in higher-dimension and higher-rank Lie groups. Thus far, we do not have any results in this direction, but we conclude the main text of this paper with an informal survey of the subject of Selberg zeta functions for groups other than $\mathrm{SL}_2$ as it currently exists, with comments on where the Artin formalism should fit into the overall picture.

For all groups of rank 1, the Artin Formalism should be within reach, thanks to the extensive work of [10] and [11] on the Selberg zeta function in the general rank one case, based on their earlier extensive work on the trace formula. Indeed, in [10], Gangolli defines a zeta function of Selberg type, along the same lines as in this article, for arbitrary $G$ of rank one and finite-dimensional representation $\chi$, but only for *co-compact* lattice $\Gamma$. Gangolli undertakes a deep study of the locations and orders of the various types of poles and zeros of this function, and he lists and proves ten properties of $Z_{\Gamma,\chi}$. The only reason, it seems, that he does not include the Artin formalism, as an eleventh, is that in the article, he does not explore the relations within of families of $Z_{\Gamma,\chi}$ as $\Gamma$, varies over the lattices in a fixed Lie group.. It would seem the proof in this case is exactly the same as that given in this paper, but no one has even stated the Artin Formalism because the result is principally of interest in the general cofinite ($G/\Gamma$-with-cusps) case. In the general $\Gamma$-cofinite case, Gangolli and Warner, in [11], use an analogous definition for $\mathrm{d}\log Z_\Gamma$, but the task of meromorphic continuation of $\mathrm{d}\log Z_\Gamma$ is made considerably more complicated by the presence of terms in the trace formula analogous to our $\tilde{I}^{\mathrm{cus}}$ and $I^{\mathrm{nce}}$. Apparently not for any essential reason, but just in order to make the calculations more comprehensible, they restrict themselves to the $\chi = \mathbf{1}$—*i.e.*, scalar—case. Therefore, in order to even state the Artin Formalism for the trace formula, or its images, such as $\mathrm{d}\log Z_{\Gamma,\chi}$, one would have to generalize the definitions and arguments there to allow for sections vector bundles associated to finite-dimensional unitary representations. In this regard, the work of Friedman in [8] in the special case of $\mathrm{SL}_2(\mathbf{C})$, locally isomorphic to $\mathrm{SO}(3,1)$, in generalizing the work of [6] to the vector case, is encouraging. In short, it looks likely that, once the technical hurdle of defining the Eisenstein kernels corresponding to singular cusps and checking that the usual spectral expansion and regularization carry through, the inductivity of the trace formula should follow in exactly the same way as in this paper. The Artin formalism itself should follow from the analogue of Proposition 19 in that setting.

The prospects are not nearly so clear for *higher rank* groups. Even, say, for a rank-two group, such as $\mathrm{SL}_3$, there is, as yet, no settled definition of a Selberg zeta function, *as a meromorphic function with symmetry relations generalizing the functional equation*. This is in sharp contrast to the situation for (nearly) all rank-one cases, thanks to the detailed study of [11]. We are aware of recent steps, as yet unpublished, towards a general theory of zeta functions of Selberg type by Jorgenson-Lang and, separately, by Stanton-Moscovici. The main difficulty in generalizing the theory, at this point, seems to be the lack of a sufficiently clear picture of the geometric side of the trace formula associated to a discrete subgroup $\Gamma$ of a real Lie group $G$. (Contrast this to the extremely well developed theory of the adelic trace formula exposed in, *e.g.*, [1], and the references contained therein.) It may be hoped that, in the near future, work on a more refined Selberg trace formula will allow the creation of a perspective unifying these disparate approaches and defining a zeta function that deserves to be called *the generalization of the Selberg zeta to arithmetic quotients of higher rank*. The naturalness of the argument for inductivity that we exhibit above argues in favor of including the *inductivity property* (or 'Artin Formalism') in a list of canonical properties to be expected of this higher-rank spectral zeta function.

## A Proof of Proposition 5

We deduce Proposition 5 from the somewhat more general Proposition 29. This proposition is stated stated in more generality than needed, and could be stated in even more general terms of abstract groups actions on sets. The present level of generality we will consider is that of $\Gamma' \subseteq \Gamma$ co-finite Kleinian groups satisfying the finite index condition $[\Gamma : \Gamma'] = n$ and acting on $\mathbf{P}^1(\mathbf{C})$, the boundary of $\mathbf{H}^3$, by fractional linear transformations.

We will denote by

$$\mathcal{P} \text{ an arbitrary non-empty set of } \mathbf{P}^1(\mathbf{C}). \tag{A.1}$$

We denote by $\mathcal{P}(\mathbf{P}^1(\mathbf{C}))$ the power set of $\mathbf{P}^1(\mathbf{C})$, that is, the set of all the subsets of $\mathbf{P}^1(\mathbf{C})$. The action of $\Gamma$ on $\mathbf{P}^1(\mathbf{C})$ can be interpreted as an action of $\Gamma$ on $\mathcal{P}(\mathbf{P}^1(\mathbf{C}))$—obviously very far from transitive! Introducing this action provides a natural framework for the following notions. For $\mathcal{P}$ as in (A.1), set

$$\Gamma_{\mathcal{P}} = \{\gamma \in \Gamma \mid \gamma\mathcal{P} = \mathcal{P}\},$$

*i.e.*, the stabilizer of $\mathcal{P}$ in $\mathcal{P}(\mathbf{P}^1(\mathbf{C}))$, as opposed the pointwise stabilizer. Similarly, denote by

$$\Gamma\mathcal{P} \text{ the orbit of } \mathcal{P} \text{ in } \mathcal{P}(\mathbf{P}^1(\mathbf{C})), \text{ under } \Gamma.$$

The same notions apply to $\Gamma'$ in place of $\Gamma$.

The action of $\Gamma$ on $\mathcal{P}(\mathbf{P}^1(\mathbf{C}))$ induces an obvious group action of $\Gamma'$ on the orbit $\Gamma\mathcal{P}$. It is obvious that $\Gamma\mathcal{P}$ (under the action of $\Gamma'$ on $\mathcal{P}(\mathbf{P}^1(\mathbf{C}))$) is the union of $\Gamma'$-orbits, indeed, of finitely many, say $h \leq n$, $\Gamma'$-orbits. With this action in mind, denote

$$\{\mathcal{P}_\alpha\}_{\alpha=1}^h \text{ a fixed system of representatives for the } \Gamma'\text{-orbits in } \Gamma\mathcal{P}. \qquad \text{(A.2)}$$

Fixing the $\{\mathcal{P}_\alpha\}$ involves making a choice of orbit representatives, which we will be assume to have been chosen and fixed once and for all for the remainder of this section. Analogous to the stabilizers $\Gamma_{\mathcal{P}}$ of $\mathcal{P}$ in $\Gamma$, we define the stabilizers

$$\Gamma'_{\mathcal{P}_\alpha} = \{\gamma' \in \Gamma' \mid \gamma'\mathcal{P}_\alpha = \mathcal{P}_\alpha\}$$

Next, fix elements $\sigma_{\mathcal{P},\alpha} \in \Gamma$, for $\alpha$ ranging from 1 to $h$, such that

$$\sigma_{\mathcal{P},\alpha}\mathcal{P} = \mathcal{P}_\alpha.$$

Our basic proposition, whose proof is very simple, concerns the ability to choose a set of coset representatives $\{\gamma_i\}_{i=1}^n$ for the coset space $\Gamma'$ "compatible" with the choice of $\left\{\mathcal{P}, \{\sigma_{\mathcal{P},\alpha}\}_{\alpha=1}^h\right\}$ and the properties of such a "compatible" set.

**Proposition 29** *With the notation as above, for each $\alpha \in \{1, \dots h\}$, let $\beta_{\alpha,j}$, $j = 1, \dots n_\alpha$, say, be a system of representatives for the coset space $\Gamma'_{\mathcal{P}_\alpha}\backslash\Gamma_{\mathcal{P}_\alpha}$, where $n_\alpha$ denotes the index $[\Gamma_{\mathcal{P}_\alpha} : \Gamma'_{\mathcal{P}_\alpha}]$. Then we have a disjoint decomposition*

$$\Gamma = \bigcup_{\alpha=1}^{h}\bigcup_{j=1}^{n_\alpha} \Gamma'\beta_{\alpha,j}\sigma_\alpha.$$

*Consequently, we have the equality*

$$n = \sum_{\alpha=1}^{h} n_\alpha.$$

*Proof.* Let $\gamma \in \Gamma$ be given. The argument proceeds in two steps. In the first step, we consider $\gamma\mathcal{P}$. There is a uniquely determined $\alpha \in \{1, \dots, h\}$ such that

$$\gamma\mathcal{P} = \sigma_\alpha\mathcal{P} = \mathcal{P}_\alpha.$$

Therefore, $\gamma \in \Gamma_{P_\alpha}\sigma_\alpha$, which is to say that

$$\text{There is an } \eta \in \Gamma_{P_\alpha} \text{ such that } \gamma = \eta\sigma_\alpha.$$

In the second step, we note that there is a uniquely determined $j \in \{1, \dots, n_\alpha\}$ such that $\eta \in \Gamma'_{P_\alpha}\beta_{\alpha,j}$.

Substituting, we see that $\gamma \in {\Gamma'}_{P_\alpha} \beta_{\alpha,j}\sigma_\alpha$. Since $\gamma$ was chosen arbitrarily in $\Gamma$, this shows that the union of the cosets $\Gamma'\beta_{\alpha,j}\sigma_\alpha$ is all of $\Gamma$. The uniqueness of the choice in each of the two steps insures that the cosets are distinct, thus disjoint. □

We obtain Proposition 5 and Corollary 6 by specializing Proposition 5 to the case of $\mathcal{P} = \{a\}$, where $a \in \mathbf{P}^1(\mathbf{C})$ is a cusp of $\Gamma$. Then the $\mathcal{P}_\alpha$ are the $b \in C_a(\Gamma')$, *i.e.*, the cusps $b$ of $\Gamma'$ such that $b|a$, and Proposition 5 follows immediately.

**Remark.** Corollary 6 gives an alternate and quick proof of the fact that $\mathrm{Cusps}(\Gamma) \subseteq \mathrm{Cusps}(\Gamma')$. For, by definition, $a \in \mathrm{Cusps}(\Gamma)$ means that $\Gamma_a^U$ is free abelian of rank 2. Then the corollary implies that the index $n_a = [\Gamma_a : \Gamma'_a] \leq n < \infty$. So ${\Gamma'}_a^U$ is also free abelian of rank 2, and $a$ is a cusp of $\Gamma'$.

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